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EXTENSION OF LOTKA – VOLTERRA MODEL AS A MEANS FOR ECONOMIC POLICY ADJUSTMENT CAUSED BY CONFLICT

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Current paper is devoted to the study of the extension of Lotka-Volterra model as a means for the adjustment of countries economic policies acting in the region of their interests. As the model we are considering a system of differential equations. The study of output dynamics based on ordinary differential equation is of great interest in nonlinear systems. The research implemented recently showed a possibility to apply Lotka-Volterra equation in economics, population dynamics, output dynamics, for the estimation of economic cost of the conflict economic. Recently the generalization of the Lotka-Volterra model studied two interactions which captured by the terms presenting outputs. The equilibrium points and conditions of the stability of Lotka-Volterra model are determined. The general solution of the system of ordinary differential equations is given.

Being interested in the study of the application for real world an example of the simulation of Lotka-Volterra model for five countries is given. The cooperative game is considered between acting countries. Through the Shapley vector of the cooperative game the constant coefficients assessing interactions between countries are determined. The effect of countries on the rate of GDP change is analyzing and is comparing with other countries.

Key words: *Lotka-Volterra model, economic policy, adjustment, cooperative game, system of ordinary differential equations, equilibrium, staibility, conflict*

1.Introduction. Lottka-Volterra model: its extensions and applications.

The Lotka-Volterra model (LVM) as a pair of two differential equations has been proposed in 1925 by American statistician Alfred J. Lotka and Italian mathematician Vito Volterra. In 2016 LVM has been extended by Devireddy L.[1], where the author presents generalized form of LVM considering the system of differential equations including diverse interactions between the species. Author considers the system of differential equations containing constant coefficients which express interactions between species. Thus, assuming that constant coefficients are parameters, it is possible to evaluate the interaction between pairs of species. Author avoids doing any assumptions related to the coefficients. However, he suggests that through the assessment of constant coefficients it is possible to assess the rate of population dynamics as the effect of one specie to the population of another specie depending of the value and sign of the coefficient (negative or positive).

There are a few studies considering the behavior of the solution of Lotka-Volterra model. The paper after Abadi,, D. Savitri,, and Ch Ummah[2] determines

[1] Devireddy L. Extending the Lotka-Volterra Equations, (2016): http://sites.math. washington.edu/~morrow/336_16/2016papers/lalith.pdf, pp.1-10

[2] Abadi, D. Savitri and Ch. Ummah, (2013): Stability analysis of Lotka-Volterra model with Holling type II functional response, Scientific Research Journal (SCIRJ), Vol I, Issue V, pp. 22-26.

the equilibrium points of the system of equations of LVM and gives the information about stability of the solutions.

Recently, population change and dynamics due to both forced and deteriorating political, economic and sociological conditions has been studied in various researches based on the LVM. Palomba was the first Palomba G.[3] to use the Lotka-Volterra equations in economics through the understanding that "business cycle is a phenomenon endogenous to the economic system". Farther, he suggested that coefficients of LVM are time dependent. Considering country's outcomes as variables Palomba showed "The interest derives from the fact that it is a model of cycles in growth rates and hence represents a step towards more realistic interpretations of cycles and growth".

The application of LVM in economics has been implemented by Solomon S[4]. The study of the concurrence in fund market implemented by Modis T.[5] and Liu H.[6]. Further, for the study of technological competition are devoted researches of Ganguly, S., Neogi, U., Chakrabarti, A.S., Chakraborti, A.[7], Wei, T., Li, Y., Song, C.[8]. The problem of marketing based on the LVM studied by Guidolin, M., Guseo, R., Mortarino, C.[9] and by Kaszkurewicz, E.,Bhaya, A.[10]. Application of LVM in trade relations studied by Wang, Z., Zhu, H. (2016)[11]. The simulation of political behavior using LVM for the class of military conflicts has been implemented by Nam T. (2006)[12], and Marasco, A., Romano, A. (2018)[13].

Relations between GDP and population dynamics. Currently, there are numerous researches devoted to the study of both relations between GDP and population dynamics as well the GDP growth and population dynamics pressure on conflicts. Authors Diehl P., Gleditsch N.P., eds (2001)[14] are studying the role of environmental degradation. Through the presentation of results of empirical research authors are examining environmental factors which may influ-

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[3] Palomba G. (2008) Lotka-Volterra Equations, Rendiconti Lincei - Scienze Fisiche e Naturali 19,(4), pp.347-357.

[4] Solomon, S. (2000), Generalized Lotka-Volterra (GLV) models of stock markets. Appl. Simul. Soc. Sci., 3,301–322.

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[6] Liu, H.C. (2020) When one stock share is a biological individual: A stylized simulation of the population dynamics in an order-driven market.Rivista di Matematica per le Scienze Economiche e Sociali , 43(2–3).

[7] Ganguly, S., Neogi, U.; Chakrabarti, A.S.; Chakraborti, A. (2017): Reaction-Diffusion Equations with Applications to Economic Systems. In Econophysics and Sociophysics: Recent Progress and Future Directions; Springer: Cham, Switzerland, pp. 131–144.

[8] Wei, T.; Li, Y.; Song, C. (2013) The Competition Model of High-Tech Industry Clusters with Limited Innovation Resources on Lotka-Volterra Model. In 2013 International Conference on Information System and Engineering Management; IEEE: Washington, DC, USA, pp. 741–744.

[9] Guidolin, M.; Guseo, R.; Mortarino, C. (2019) Regular and promotional sales in new product life cycles: Competition and forecasting. Comput. Ind. Eng., 130, pp.250–257

[10] Kaszkurewicz, E.; Bhaya, A. (2018,) Modeling market share dynamics under advertising effort and word-of-mouth interactions between customers. IEEE Trans. Comput. Soc. Syst., 5, 382–390.

[11] Wang, Z.; Zhu, H. (2016) Testing the trade relationships between China, Singapore, Malaysia and Thailand using Grey Lotka-Volterra competition model. Kybernetes, 45, pp.931–945.

[12] Nam, T. (2006), The broken promises of democracy: Protest-repression dynamics in Korea 1990–1991. Mobilization 11, pp.427–442

[13] Marasco, A., Romano, A. (2018) Deterministic modeling in scenario forecasting: Estimating the effects of two public policies on intergenerational conflict. Qual. Quant., 52, pp.2345–2371

[14] Diehl, Paul F, Nils P. Gleditsch, eds, (2001) Environmental Conflict. Oxford, https://www.prio.org/Publications/Publication/?x=667

ence state decisions to avoid violence, and future conflicts prevention. The research after Goldstone, J, E Kaufmann and M Toft (2012)[15] is devoted to the study of the effect of population increase on social conflict. Authors estimations imply that the average change in population for studied sample of countries caused roughly 4.2 additional years of full-blown civil war in the 1980s relative to the 1940s.

Interstate conflicts, civil war and socio-economic (economic growth, per capita GDP) and demographic (death of combats and civilian population, forced migration) consequences are studied by Melander E., Öberg M.(2004)[16]. In addition, they presented new measures of armed conflicts and gave magnitude and scope of fighting. The central point of view for this research is the large variation in forced migrant flows across countries experiencing armed conflict. The conflict between Armenia and Azerbaijan in the late of twenty century considered as the case study. According to this research countries (Armenia and Azerbaijan) implemented migrants exchange such that the population exchange dynamics was equal to zero. However, 2020 war affected deaths of combats and civilian population from both sides and the migration of Armenian population from Karabakh. The work Acemoglu D., Fergusson L., (2020)[17] suggests that measures supporting GDP growth in some countries are significant for both to provide welfare benefits and enabling better adaptation for the population.

The paper after Puliafito S.E., Grand M.C (2008)[18] proposes a model based on LVM to describe relation between population dynamics and gross domestic product. This model presents a new and simple “conceptual explanation of the interactions and feedbacks among the principal driving forces leading to the present transition. The estimated results for the temporal evolution of world population, gross domestic product, primary energy consumption and carbon emissions are calculated from year 1850 to year 2150. The calculated scenarios are in good agreement with common world data and projections for the next 100 years.” Thus, this paper suggests relations between population dynamics and GDP growth based on the LVM.

The problem of the relationship between population and economic growth is an essential issue both for developed and developing countries. The article after Thuku G.K., Paul G. and Almadi O.(2013)[19] considers the vector autoregression (VAR) model including GDP and population as variables. Authors implemented an experimental research and found that the relationship between population and economic growth is bi-directional; proposed VAR model gives the relationship between population and economic growth for Kenya.

[15] Goldstone, J, E Kaufmann and M Toft (2013), Political Demography: How Population Changes are Reshaping International Security and National Politics, Oxford: Oxford University Press, 20 March, pp.183

[16] Melander E., Öberg M. (2004), Forced Migration: The Effects of the Magnitude and Scope of Fighting Uppsala Peace Research Papers No. 8 , Department of Peace and Conflict Research

[17] Acemoglu D., Fergusson L., (2020): Population and Conflict. Review of Economic Studies 87, pp.1565–1604

[18] Puliafito S.E., Grand M.C., Modeling population dynamics and economic growth as competing species:An application to CO2 global emissions, http://people.kzoo.edu/barth/math280/articles/co2emissions.pdf

[19] Thuku G.K.,Paul G. and Almadi O. (2013)The impact of population change on economic growth in Kenya, International Journal of Economics and Management Sciences, Vol. 2, No. 6, pp. 43-60.

The paper after Kitov I.O.(2006)[20] is devoted to the study of the relationship between GDP growth rate and the population given as:

$$\frac{dG(t)}{G(t)}dt = 0.5\frac{dN(t)}{N(t)} + 1/T_{cr}(t),$$

where $G(t)$ is the real GDP as a function of time, $N(t)$ is the number of people with the specific age at time t. Author has presented an empirical model on the dependence between output growth and population. Results of the paper allow substantiating the study of population dynamics rate through the study of output growth.

In contrast to the well-known LVM model, the author Ditzen J.(2017)[21] considers two countries one of which is prey and another is predictor. Countries outputs are considered as variables. According to authors assumption if per capita income of one country increases than other country's output increases also. Thus, the proposed LVM model examines the dependence of the rate of output depending from the interactions between countries expressed by their outputs.

Summarizing researches of LVM we could conclude the following. There are studies that are devoted to the extension, generalization, applications in economics, interstates conflicts, and use GDP as variables instead population in contrast to the population given in recently developed LVM.

We present an extended LVM concept of the interstate conflict relations problem's simulation in current study. As LVM we will consider an extending of LVM given by Devireddy L. (2016)[22]. Instead of species we will consider five countries and instead of the population as the variable characterizing the country we will follow the model given in the paper after Ditzen J.(2017)[23] and as variables characterizing countries we will consider outputs of countries. Consequently, we will consider an interstate relations behavior based on the GDP dynamics rate expressed by the interactions between countries. Constant coefficients used in the simulation model are defined through the population of interacting pairs of countries. The extending LVM studied in current paper allows presenting the simulation based both on GDP dynamics and population growth. Thus, we have fulfilled the requirement to consider broader factors for the simulation of interstate relations, in particular GDP and constant coefficients of interstate interactions expressed through the population dynamics.

To value the interactions between countries we consider country's GDP as

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[20] Kitov I.O. (2006,) GDP growth rate and population, 2006, p.60 https://www.researchgate.net/publication/5127548_GDP_Growth_Rate_and_Population

[21] Ditzen J.Cross Country Convergence in a General Lotka-Volterra Model, (2017):Journal of Special Economic Analysis. Centre for Energy Economics Research and Policy (CEERP) and Spatial Economics and Econometrics Centre (SEEC), Heriot-Watt University, Edinburgh, UK, December 1, p.21. https://pure.hw.ac.uk/ws/portalfiles/portal/16264686/CrossCountryConvergenceInLVM.pdf

[22] Devireddy L. Extending the Lotka-Volterra Equations, (2016): http://sites.math.washington.edu/~morrow/336_16/2016papers/lalith.pdf, pp.1-10

[23] Ditzen J.Cross Country Convergence in a General Lotka-Volterra Model, (2017):Journal of Special Economic Analysis. Centre for Energy Economics Research and Policy (CEERP) and Spatial Economics and Econometrics Centre (SEEC), Heriot-Watt University, Edinburgh, UK, December 1, p.21. https://pure.hw.ac.uk/ws/portalfiles/portal/16264686/CrossCountryConvergenceInLVM.pdf

the assessment of country's state, and the interaction between countries. Consequently, country's GDP has same meaning as species population in Lotka-Volterra model.

Thus, we considered a system of five countries as the simulation model as follows

I) countries 1 and 2 are conflicting with each other,

ii) countries 3,4, and 5 are interacting separately through the cooperation with countries 1 and 2,

Farther, we studied equilibrium points and behavior of outputs of interacting countries.

2. Definitions and notations. Papers after Devireddy L. (2016)[24] and Marasco A., Picucci A. Romano A.(2016)[25] proposed an extension of Lotka-Volterra model

$$\frac{dp_i}{dt} = g_i p_i(t) + \sum_{1 \le i,i,j \le n, i \ne j} d_{i,j} p_i(t) p_j(t) \text{, where} \tag{1}$$

i) $p_i(t)$ is the population of $i-th$ species,

ii) $g_i p_i(t)$ represents either the growth or natural death rate of the species and simultaneously it is proportional to the species population,

iii) $d_{i,j}$ is a constant representing $j-th$ contribution to assess the effect of $d_{i,j} p_i(t) p_j(t)$ to the differential $\frac{dp_i(t)}{dt}$ which is the measure of the rate of change of $i-th$ population at time t following after the interaction with $j-th$ specie.

iv) we will assume that two species are interacting to benefit which other. Depending from the sign of the coefficient $d_{i,j}$ the rate $d_{i,j} p_i(t) p_j(t)$ could be positive or negative. In first case the contribution to the $\frac{dp_i(t)}{dt}$ is positive from the interaction between $i-th$ and $j-th$ species. The negative sign of the coefficient $d_{i,j}$ shows that the $j-th$ specie causes the decrease of the population of $i-th$ specie from the interaction with $j-th$ specie.

Following of Ditzen J.(2017)[26] we will consider "two species as countries R and P. Assume that y_R and y_P are outputs of countries R and P correspondently. Further, following to LVM model, "one is preying upon the other.

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[24] Devireddy L. Extending the Lotka-Volterra Equations, (2016): http://sites.math. washington.edu/~morrow/336_16/2016papers/lalith.pdf, pp.1-10

[25] Marasco A., Picucci A, Romano A. (2016) Market share dynamics using Lotka–Volterra models, Technological Forecasting and Social Change, Volume 105, April, pp. 49-62

[26] Ditzen J.Cross Country Convergence in a General Lotka-Volterra Model, (2017):Journal of Special Economic Analysis. Centre for Energy Economics Research and Policy (CEERP) and Spatial Economics and Econometrics Centre (SEEC), Heriot-Watt University, Edinburgh, UK, December 1, p.21. https://pure.hw.ac.uk/ws/portalfiles/portal/16264686/CrossCountryConvergenceInLVM.pdf

The two interactions can be captured by the terms $\rho_{PR} y_R y_P$ and $\rho_{RP} y_R y_P$. Consequently similarly to the LVM the system of equations expressed by" Ditzen J.(2017)[27] is as follows:

$$\frac{dy_R}{dt} = \alpha_R y_R + \rho_{PR} y_R y_P \qquad (2)$$

$$\frac{dy_P}{dt} = \alpha_P y_P + \rho_{RP} y_R y_P \text{"} \qquad (3)$$

" Then Lotka-Voterra model can be applied to a more general setting, as for example interactions between countries. In addition, the model is not limited to two countries or equations. In the case of N countries, there would be N difference equations. Still, it would be possible that some countries benefit, while others lose from the interactions. Samuelson describes such an extension with n > 2 predators and prey".

Everywhere following Ditzen J.(2017)[28] we will assume that the system of countries is considering as species proposed in Lottka-Volterra model. Countries GDPs are given similarly to the population of Lottka - Volterra model denoted as $p_i(t), i = 1,2,...,n$ and we followed Arakelyan A., Makaryan L. (2021)[29] approach. Let us consider the system of ordinary differential equations as the example of (1) as follows:

$$\frac{dGDP_l}{dt} = g_l GDP_l(t) + \sum_{,j=1,j\neq l}^{5} d_{1,j} GDP_1(t) GDP_j(t), \text{ where } l = 1,2,...,5: \quad (4)$$

Assume that $t \in [0,T]$, $GDP_k(0) = GDP_{k0}, k = 1,2,...,5,$ where as we proposed $GDP_k(t)$ is output of $k-th$ country, $k = 1,2,...,5$.

Analysis of five countries model (4). The solution of the model (4) requires to find equilibria as a mean based on the solution of the system of equations:

$$\frac{dGDP_l}{dt} = g_l GDP_l(t) + \sum_{j=1,j\neq l}^{5} d_{l,j} GDP_l(t) GDP_j(t) = 0, \text{where } l = 1,2,...,5. \quad (5)$$

From the system of equations (5) follows that

$$GDP_l(g_l + \sum_{j\neq l,j=1}^{5} d_{l,j} GDP_l) = 0, \text{ where } l = 1,2,...,5. \qquad (6)$$

Consequently, this system of the equations has the solution $(y_1, y_2, y_3, y_4, y_5) = (0,0,0,0,0)$ which is one of equilibrium positions.

Let's consider the system of equations:

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[27] Ditzen J.Cross Country Convergence in a General Lotka-Volterra Model, (2017):Journal of Special Economic Analysis. Centre for Energy Economics Research and Policy (CEERP) and Spatial Economics and Econometrics Centre (SEEC), Heriot-Watt University, Edinburgh, UK, December 1, p.21. https://pure.hw.ac.uk/ws/portalfiles/portal/16264686/CrossCountryConvergence InLVM.pdf

[28] Ditzen J.Cross Country Convergence in a General Lotka-Volterra Model, (2017):Journal of Special Economic Analysis. Centre for Energy Economics Research and Policy (CEERP) and Spatial Economics and Econometrics Centre (SEEC), Heriot-Watt University, Edinburgh, UK, December 1, p.21. https://pure.hw.ac.uk/ws/portalfiles/portal/16264686/CrossCountryConvergence InLVM.pdf

[29] Arakelyan A., Makaryan L. (2021): Model of territorial conflict and international military cooperation, Journal oof Yerevan University, Economics, N3(36), pp.56-72.

$$0GDP_l + \sum_{j \neq l, j=1}^{5} d_{j,l} GDP_j = -g_l \qquad (7)$$

where $l = 1,2,3,4,5$ and assume that the determinant Δ of the system of the equations (7) isn't equal to zero. Consequently, this system has unique solution $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5)$. The system of the equations (6) has two solutions which are simultaneously equilibrium points of the system of ordinary differential equations.

Let us denote

$$F_k(y_1, y_2, y_3, y_4, y_5) = g_k y_k(t) + \sum_{1 \le i,, j \le 5, j \neq k} d_{k,j} y_k(t) y_j(t), \qquad (8)$$

where $k = 1,2,\ldots,5$.

3. Linearization of the system (6). Let us denote the equilibrium point of the system of ordinary differential equations (4) as $(x_1, x_2, x_3, x_4, x_5)$, where $(x_1, x_2, x_3, x_4, x_5)$ equal to $(0,0,0,0,0)$ or $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5)$. Consider the nature of the behavior of variables at some small deviation of the system from equilibrium position. We introduce new variables $\xi_i, i = 1,2,3,4,5$ defined as displacement relative to equilibrium point $(x_1, x_2, x_3, x_4, x_5)$:

$$GDP_i = x_i + \xi_i, \text{ where } i = 1,2,3,4,5 \qquad (9)$$

Substituting expressions (9) in (4) we will get:

$$\frac{dx_k}{dt} + \frac{d\xi_k}{dt} = g_k(x_k + \xi_k) + \sum_{1 \le i,k,j \le 5, k \neq j} d_{k,j}(GDP_k + \xi_k)(GDP_j + \xi_j), \qquad (10)$$

where $k = 1,2,\ldots,5$.

Since $(x_1, x_2, x_3, x_4, x_5)$ is an equilibrium point than $\frac{dx_k}{dt} = 0,\ k = 1,2,\ldots,5$. Consequently, from (10) follows:

$$\frac{d\xi_k}{dt} = g_k(x_k + \xi_k) + \sum_{1 \le i,k,j \le 5, k \neq j} d_{k,j}(x_k + \xi_k)(x_j + \xi_j), \text{where } k = 1,2,\ldots,5. \qquad (11)$$

Denote the set of countries as $S = \{1,2,3,4,5\}$. Since

$F_k(y_1, y_2, y_3, y_4, y_5) = g_k y_k(t) + \sum_{1 \le i,, j \le 5, j \neq k} d_{k,j} y_k(t) y_j(t)$, where $k \in S,$ is continuous, differentiable relative to y_1, y_2, y_3, y_4, y_5, than it has derivatives in equilibrium point as follows:

(i) relative to y_k $F_{k,y_{k1}}\big|_x = (g_k + d_{k,l}x_l + d_{k,m}x_m + d_{k,n}x_n + d_{k,p}x_p)$,

$F_{k,y_l}\big|_x = d_{k,l}x_k$, $F_{k,y_m}\big|_x = d_{k,m}x_k$, $F_{k,y_n}\big|_x = d_{k,n}x_k$, $F_{k,y_p}\big|_x = d_{k,p}x_k$,

where $l, m, n, p \in S \setminus \{k\}, k \in S.$

Since $(x_1, x_2, x_3, x_4, x_5)$ is an equilibrium point of the system of ordinary differential equations (4) than $F_k(x_1, x_2, x_3, x_4, x_5) = 0$, where $k = 1,2,...,5$.

Therefore, since functions $F_k(x_1, x_2, x_3, x_4, x_5)$, where $k = 1,2,...,5$ are continuous and have derivatives at the equilibrium point, we could decompose as Taylor series. Consequently, using only first order derivatives of the functions $F_k(x_1, x_2, x_3, x_4, x_5)$, where $k = 1,2,...,5$ we could present functions as Taylor series decompositions relative to $\xi = (\xi_1, \xi_2, \xi_3, \xi_4 \xi_5)$ as follows:

$$F_l(x_1, x_2, x_3, x_4, x_5) = (g_l + \sum_{j=1, j\neq l}^{5} d_{l,j} x_j) + \sum_{j\neq l, j\equiv 1}^{5} d_{l,j} x_l \xi_j \text{, where } l = 1,2,...,5 \quad (12)$$

The system of equations (12) allows the presentation of the system of ordinary differential equations (4) as linear as follows:

$$\frac{d\xi_l}{dt} = (g_l + \sum_{j=1, j\neq l}^{5} d_{l,j} x_j) + \sum_{j\neq l, j\equiv 1}^{5} d_{l,j} x_l \xi_j \text{, where } l = 1,2,...,5. \quad (13)$$

Eigenvalues definition

Case 1. Equilibrium point $(x_1, x_2, x_3, x_4, x_5)$ equal to $(0,0,0,0,0)$. Consequently, the system (13) has solutions $g_i - \lambda_i$, $i = 1,2,...,5$.

Case 2. Equilibrium point $(x_1, x_2, x_3, x_4, x_5)$ equal to $(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5)$. Consequently, the system (13) admits the representation as follows:

$$\frac{d\xi_1}{dt} = (g_l + \sum_{j\neq l, j\equiv 1}^{5} d_{l,j} \bar{y}_j)\xi_l + \sum_{j\neq l, j=1}^{5} d_{l,l} \bar{p}_l \xi_j, \qquad l = 1,2,...,5 \quad (14)$$

Denote as $\alpha_k = (\alpha_{k1}, \alpha_{k2}, \alpha_{k3}, \alpha_{k4}, \alpha_{k5})$ eigenvector corresponding to eigenvalue λ_k, $k = 1,2,...,5$. Thus, the general solution of the system (5) in vector form is

$$GDP(t) = (GDP_1(t), GDP_2(t), GDP_3(t), GDP_4(t), GDP_5(t))^{\mathrm{T}} =$$

$$(\sum_{i=1}^{5} C_i \exp(\lambda_{it})\alpha_k)_{k=1,2,3,4,5}, \quad (15)$$

where $\alpha_k = (\alpha_{k1}, \alpha_{k2}, \alpha_{k3}, \alpha_{k4}, \alpha_{k5}), . k = 1,2,...,5$.

4. Simulation model. The simulation model presented below assumes that countries 1,2,3,4,5 are denoted as B1,B2,B3,B4,B5.

The statement of the conflict. The simulation model assumes the existence of relations between countries as follows. According the similarity with the LVM model we assume that country B1 is a prey and the country B2 is predator upon the country B1. Therefore, the constant $d_{1,2}$ is assumed negative in the simulation model. Countries B3, B4, B5 are intermediate actors and are acting to adjust relations between countries B1 and B2. Nevertheless, they have own economic interests in the results of the conflict resolution. The countries B3 and B4 are cooperating with country B1 through the participation in peaceful process of negotiations. In addition, they are directing the country B1 for the benefit. In case country B5 pursues the opposite policy, it hurts the country B2 for the success in the relations with B1 and other countries.

Each country has three choices {1,2,3} as strategies, and depending from these strategies country's payoff weights are changing in the segment [0,10] and are given in the table 1 (Appendix).

Using payoffs weights given in the table 1 define the cooperative game between players $\{i, j, k\}$, where $i, j, k \in \{1,2,3,4,5,\}$ with characteristic function as follows.

Denote the weight as $w_{l,q}$, where l is the country $l \in \{1,2,3,4,5,\}$, q is the strategy, $q \in \{1,2,3\}$.

(i) $v(l) = \underbrace{\max w_{lq}}_{q\in\{1,2,3\}}$, $l \in \{1,2,3,4,5\}$. (16)

(ii) $v(l,p) = \underbrace{\max w_{lq}}_{q\in\{1,2,3\}} + \underbrace{\max w_{pq}}_{q\in\{1,2,3\}}$, $l, p \in \{1,2,3,4,5\}$. (17)

(iii $v(i,j,k) = \underbrace{\max w_{iq}}_{q\in\{1,2,3\}} + \underbrace{\max w_{jq}}_{q\in\{1,2,3\}} + \underbrace{\max w_{kq}}_{q\in\{1,2,3\}}$ $i, j, k \in \{1,2,3,4,5\}$. (18)

Denote the game between players $\{i, j, k\}$, as:

$$G =< v(S), S \subseteq \{i, j, k\} >, \quad (19)$$

where $v(S), S \subseteq \{i, jk\}$ is characteristic function defined by (16)-(18).
Let us define games as follows:

(i) for countries {B3,B4,B5} as follows:
$v(B3) = v(B4) = v(B5) = 10, v(B3,B4) = 20, v(B3,B5) = 20, v(B4,B5) = 20,$
$v(B3,B4,B5) = 30.$

(ii) for countries {B1,B3,B4} as follows:
$v(B1) = 5, v(B3) = v(B4) = 10, v(B3,B4) = 20, v(B1,B3) = v(B4,B1) = 15,$
$v(B1,B3,B4) = 25.$

(iii) for counties {B2, B3, B4} as follows:
$v(B2) = v(B3) = v(B4) = 10, v(B3,B4) = 20, v(B2,B3) = 20, v(B4,B2) = 20,$
$v(B2,B3,B4) = 30.$

(iv) for countries {B1,B2,B5} as follows:
$v(B1) = 5, v(B2) = v(B5) = 10, v(B1,B2) = v(B1,B5) = 15, v(B2,B5) = 20,$
$v(B1,B2,B5) = 25.$

Define Shapley vectors given in Shapley L.S. (1988)[30] for these games as follows . Assume that the set of players is N={1,2,…,n} and the characteristic function is $v(S)$, where $S \subseteq N$. Than Shapley vector is as follows:

[30] Shapley L.S. (1988), A value for n-person games/ in: The Shapley value, Essays in honor, of Lloyd S. Shapley, Edited by Alvin E. Roth/Published by the Press Syndicate of the University of Cambridge, pp.31-41.

$$\Phi_i(v) = \sum_{S \subset N, i \in S} \gamma^S (v(S) - V(S/\{i\})), \tag{20}$$

where $\gamma^S = \frac{(n-|S|)!(|S|-1)!}{n!}$, $i \in \{1,2,...,n\}$.

Values of Shapley vectors based on the payoff matrix are assessing according to formula (16) - (18).

Thus, we have constant coefficients of the system of equations which is given in the Table 4 (Appendix).

Consequently, substituting the expression of the Shapley vector into the system (5) as coefficients $d_{i,j}$, $i = 1,2,3,4,5$ we are getting the system with assessments of the rates of change $d_{i,j} GDP_i(t) GDP_j(t)$, where $i = 1,2$, $j = 3,4,5$. The sign of $d_{i,j} GDP_i(t) GDP_j(t)$ $i = 1,2$ and $j = 3,4,5$ we will define depending from the requirements of simulation model and relations between countries.

We assume that $d_{i,i} = g_i$. Let us substitute values of $d_{i,j}$ into the system (5). Thus, the system (21) - (25) will be converted to the form:

$$\frac{dGDP_1}{dt} = 0.5GDP_1(t) - 1.5GDP_1(t)GDP_2(t) + 9.17GDP_1(t)GDP_3(t) + 11.7GDP_1(t)GDP_4(t) + 9.17GDP_1(t)GDP_5(t) \tag{21}$$

$$\frac{dGDP_2}{dt} = 1.5p_2(t) + 2.0GDP_1(t)GDP_2(t) + 9.17GDP_2(t)GDP_3(t) + 11.7GDP_2(t)GDP_4(t) + 9.17GDP_2(t)GDP_5(t) \tag{22}$$

$$\frac{dGDP_3}{dt} = 2.5GDP_3(t) + 9.17GDP_1(t)GDP_3(t) + 9.17GDP_3(t)DP_2(t) + 10GDP_3(t)GDP_4(t) + 7.5GDP_3(t)GDP_5(t) \tag{23}$$

$$\frac{dGDP_4}{dt} = 2.5GDP_4(t) + 11.7GDP_1(t)GDP_4(t) + 11.7GDP_4(t)GDP_2(t) + 10GDP_3(t)GDP_4(t) + 7.5GDP_4(t)GDP_5(t) \tag{24}$$

$$\frac{dGDP_5}{dt} = 2.5GDP_5(t) + 9.17GDP_1(t)GDP_5(t) + 9.17GDP_5(t)GDP_2(t) + 7.5GDP_3(t)GDP_5(t) + 7.5GDP_4(t)GDP_5(t) \tag{25}$$

From (15) follows the general solution of the system (21)-(25) as

$$GDP_k(t) = \sum_{i=1}^{5} C_i \exp(\lambda_i t)\alpha_{ki}, \; k = 1,2,...,5. \tag{26}$$

The system (21) - (25) will have an asymptotically stable solution if and only if all eigenvalues are negative. Consideration of eigenvalues shows that $\lambda_1, \lambda_3, \lambda_5$ are positive and there for the simulation model is asymptotically non stable. We note that the system will have an asymptotically stable solution if and only if all the real parts of the eigenvalues of the Jacobian are negative.

The weights matrix given in simulation model allows making changes depending from the interests of countries {3,4,5} and the impact of their activities overcomes on the state of countries {1,2} and simultaneously on their states also.

5. Analysis of the general solution per country. Analysis of the general solution per country allows to assess the rate of the change of $i-th$ country at time t following after the interaction with $j-th$ country. Representation of the system of general solution per country shows that

$$GDP_1(t) = C_1\exp(36.19t)0.87 + C_2\exp(-16.02t)(-2.16) + C_3\exp(7.14t)(-1.77) + C_4\exp(-4.23t)0.01 + C_5\exp(1.2t)(-43.92) \quad (27)$$

$$GDP_2(t) = C_1\exp(36.19t)0.99 + C_2\exp(-16.02t)(-1.66) + C_3\exp(7.14t)(-1.02) + C_4\exp(-4.23t)0.006 + C_5\exp(1.2t)41.92 \quad (28)$$

$$GDP_3(t) = C_1\exp(36.19t)1.07 + C_2\exp(-16.02t)(0.65) + C_3\exp(7.14t)(8.79) + C_4\exp(-4.23t)0.43 + C_5\exp(1.2t)(1.37) \quad (29)$$

$$GDP_4(t) = C_1\exp(36.19t)1.16 + C_2\exp(-16.02t)(1.55) + C_3\exp(7.14t)(-6.67) + C_4\exp(-4.23t)0.44 + C_5\exp(1.2t)(0.91) \quad (30)$$

$$GDP_5(t) = C_1\exp(36.19t) + C_2\exp(-16.02t) + C_3\exp(7.14t) + C_4\exp(-4.23t) + C_5\exp(1.2t) \quad (31)$$

The formula (27) suggests highest negative rate of change of the output of the country C1 at time t as a consequence of the value of summed term $C_5\exp(1.2t)(-43.92)$. This term corresponds eigenvector B5th country's component. There is an impact of 2nd and 3rd summed terms. However, these terms have little negative impact on the rate of change of the output of the country B1 comparing with $C_5\exp(1.2t)(-43.92)$ summed term.

Farther, formula (28) suggests highest positive rate of change of B2nd country's output at time t as a consequence of the value of summed term $C_5\exp(1.2t)(41.92)$. This term corresponds to eigenvector of B5th country's component. Above given suggestions follow from the efficiency rating scores and payoff functions of cooperative game {3,4,5, V(S)} determined by these scores.

Conclusion. We proposed an approach to extend Lotka-Volterra model considering as an example of five countries having interests in one economic region. Among these countries we distinguished two countries having conflicting interests. The conflict has been caused by the disorder of territorial integrity and as a consequence interacting the exhausting of economic resources of one another. The model considers also the group of countries interacting with these two distinguished countries as well having mutual interactions and interests in the region separately from distinguished two countries.

We studied researches devoted to the substantiation of the dependence between output growth and population dynamics. In addition, we used the model of Lotka – Volterra which considers the output instead of he population of species of Lotka – Volterra model. Consequently, we proposed the model of Lotka-Volterra through the use of results given in rewired papers.

The general Lotka – Volterra model is proposed and the solution of this model is given. As the outcome of considered model the simulation model as the example of five countries having interests in the region is considered. The simulation model allows doing changes on the assessment of constants coefficients using in Lotka – Volterra model.

The numerical scheme is produced to provide the solution of general model as well of simulation model. Farther, the simulation model could serve as the schema for the enhancement of the number of entities of the model, define constants coefficients valuing interactions between entities (species as considered in Lotka-Voltera model). The equilibrium point, eigenvalues and eigenvectors of the simulation model are defined and the general solution is presented.

We argue that Lotka –Volterra model allows the enhancements as means to adjust economic policy in the region as the region of conflicting relations implementing by countries having opposite interests.

ԱՐԱՄ ԱՌԱՔԵԼՅԱՆ, ԼԵՈՆ ՄԱԿԱՐՅԱՆ – *Լոտկա-Վոլտերայի մոդելի ընդլյանումը որպես կոնֆլիկտով պայմանավորված տնտեսական քաղաքականության կարգավորման միջոց* – Հոդվածում ուսումնասիրվում է Լոտկա-Վոլտերայի մոդելի ընդլայնման խնդիրը: Մոդելը ներկայացվում է դիֆերենցիալ հավասարումների համակարգի հիման վրա, ինչը հնարավորություն է տալիս ուսումնասիրելու մոդելավորվող օբյեկտը՝ կիրառելով վերջինիս վարքի հետազոտությունը դինամիկայում, քանի որ առաջարկված եղանակը մեծ հետաքրքրություն է ներկայացնում ոչ գծային համակարգերի համար: Վերջին տարիներին իրականացված հետազոտությունները հնարավորություն են տվել կիրառելու Լոտկա-Վոլտերայի հավասարումը տնտեսագիտության, բնակչության փոփոխության, արտադրանքի դինամիկաներն ուսումնասիրելիս, ինչպես նաև հակամարտությունների ընթացքում տնտեսական ծախսերը գնահատելիս: Վերջերս Լոտկա-Վոլտերայի մոդելի ընդհանրացմամբ ուսումնասիրվել են երկու երկրների արդյունքների միջև փոխազդեցությունները: Որոշվում են Լոտկա-Վոլտերայի մոդելի հավասարակշռության կետերը և կայունության պայմանները: Տրված է սովորական դիֆերենցիալ հավասարումների համակարգի ընդհանուր լուծումը: Հետաքրքրված լինելով իրական աշխարհի համար գործնական խնդիրների ուսումնասիրությամբ՝ դիտարկվում է հինգ երկրների համար Լոտկա-Վոլտերայի մոդելավորման օրինակ: Գործող երկրների միջև դիտարկվում է կոոպերատիվ խաղը: Կոոպերատիվ խաղի Շեյպլիի վեկտորի միջոցով որոշվում են երկրների միջև փոխգործակցությունը գնահատող հաստատուն գործակիցները: Երկրների ազդեցությունը ՀՆԱ-ի փոփոխության տեմպերի վրա վերլուծվում և համեմատվում է այլ երկրների հետ:

Բանալի բառեր – *Լոտկա-Վոլտերայի մոդել, տնտեսական քաղաքականություն, կարգավորում, կոոպերատիվ խաղ, սովորական դիֆերենցիալ հավասարումների համակարգ, հավասարակշռություն, կայունություն, կոնֆլիկտ*

АРАМ АРАКЕЛЯН, ЛЕОН МАКАРЯН – ***Обобщение модели Лотка-Вольтерра, как средство регулирования экономической политики, обусловленной конфликтом.*** – Настоящая статья посвящена изучению проблемы расширения модели Лотки-Вольтерры как средства корректировки экономической политики стран, действующих в регионе своих интересов. В качестве модели мы рассматриваем систему дифференциальных уравнений. Изучение динамики изменения ВВП на основе обыкновенного дифференциального уравнения представляет большой интерес для нелинейных систем. Проведенное недавно исследование показало возможность применения уравнения Лотки-Вольтерра в экономике, динамике изменения населения, динамике развития производства, с целью оценки экономической стоимости конфликта. Недавно в рамках обобщения модели Лотки-Вольтерры были изучены попарные взаимодействия между странами, которые представлялись на основе ВВП результатов. Определены точки равновесия и условия устойчивости модели Лотки-Вольтерра. Дано общее решение системы обыкновенных дифференциальных уравнений. Будучи заинтересованными в решении проблемы приложения модели Лотки-Вольтерра для реального мира приводится пример модели для пяти стран. Рассматривается кооперативная игра между действующими странами. Через вектор Шепли кооперативной игры определяются постоянные коэффициенты, оценивающие взаимодействия между странами. Влияние стран на скорость изменения ВВП анализируется и сравнивается с другими странами. Отметим, что приведенный подход позволяет изучить взаимодействие между более чем пятью странами.

Ключевые слова: *Лотка-Вольтерра, модель, экономическая политика, регулирование, кооперативная игра, система обыкновенных дифференциальных уравнений, равновесие, устойчивость, конфликт*