PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2013, № 1, p. 32–37

Mechanics

CONTACT PROBLEM FOR AN ELASTIC HALF-PLANE WITH HETEROGENEOUS (PIECEWISE-HOMOGENEOUS) ELASTIC OVERLAY IN THE PRESENCE OF SHEAR INTERLAYER

A. V. KEROBYAN*

Chair of Mechanics YSU, Armenia

The contact problem for an elastic half-plane strengthened at its boundary by heterogeneous (piecewise-homogeneous) elastic overlay consisting of two semi-infinite pieces and one finite piece having different elastic characteristics has been considered. It was supposed that the contact interaction between the deformable foundation and an overlay was realized through a shear interlayer having different physico-mechanical properties and geometric configuration. The contact triple (overlay, interlayer, half-plane) was assumed to be deformed under the action of horizontal forces applied to the overlay. Using generalized Fourier transform the determinational problem of unknown contact stresses are reduced to the systems of Fredholm's second kind integral eqations with two unknown functions within the different intervals, which in the region in the large change values of ratio of the problem characteristic parameters in the B Banach space may be solved by the method of successive approximations. Possible particular cases have been observed and the character of change contact stresses is illustrated in different contact parts.

Keywords: heterogeneous elastic overlay (stringer), generalized Fourier transform, system of integral equations, operator equation.

1. The Statement of Problem and Obtaining of Resolving Equations. Let an elastic half-plane (with the modulus of elasticity E, or the shear modulus G, the Poisson coefficient ν), the boundary of which at $y = 0$ (in xOy plane) be strengthened by small thickness h heterogeneous (piecewise-homogeneous) infinite overlay consisting of three different pieces with various elastic characteristics, has the modulus of elasticity equal to E_1 for $x > a$, equal to E_2 for $|x| < a$ and equal to E_3 for $x < -a$. The contact interaction between the half-plane and the overlay is realized by a thin layer of glue with other physico-mechanical characteristics (E_k, ν_k, h_k). The problem consists in determination of unknown contact stresses, when the horizontal forces P are applied along the overlay length in points $x = \pm b$, where $b > a$.

It is supposed that for the overlay (stringer) the model of contact along the line is realized, and for the layer of glue it is the condition of pure shear, due to which only the shear contact stresses are acting under them [1–5].

* E-mail: agas50@ysu.am

In view of the above mentioned, differential equation of overlay equilibrium written by generalized functions will have the following form [2–5]:

$$\frac{dU^{(1)}(x)}{dx}=\frac{\tau_a^+(x)}{E_1h}+\frac{\tau_a^-(x)}{E_3h}+\frac{\tau_0(x)}{E_2h}-\frac{P\delta(x-b)}{E_1h}-\frac{P\delta(x+b)}{E_3h}-$$
$$-u'_a\delta(x-a)-u'_{-a}\delta(x+a),\quad -\infty<x<\infty, \tag{1.1}$$

subject to conditions

$\left(du^{(1)}/dx\right)_{x=a-0}-\left(du^{(1)}/dx\right)_{x=a+0}=u'_a\neq 0$, $\left(du^{(1)}/dx\right)_{x=-a-0}-\left(du^{(1)}/dx\right)_{x=-a+0}=u'_{-a}\neq 0$.

Here

$$U^{(1)}(x)=\theta(-x-a)\frac{du^{(1)}}{dx}+\left[\theta(x+a)-\theta(x-a)\right]\frac{du^{(1)}}{dx}+\theta(x-a)\frac{du^{(1)}}{dx}, \tag{1.2}$$

$\tau_a^+(x)=\theta(x-a)\tau(x)$, $\tau_a^-(x)=\theta(-x-a)\tau(x)$, $\tau_0(x)=\left[\theta(x+a)-\theta(x-a)\right]\tau(x)$, $\tau(x)=\tau_a^+(x)+\tau_a^-(x)+\tau_0(x)$, $u^{(1)}(x)$ are the horizontal displacements at the overlay points, $\tau(x)$ is the intensity of tangential contact stresses, $\theta(x)$ is the Heaviside unit function, $\delta(x)$ is the δ-function, u'_a, u'_{-a} are yet unknown coefficients.

On the other hand, in case when only the tangential stresses with intensity $\tau(x)$ act on $y=0$ boundary, the deformation of boundary points of elastic half-plane is determined as

$$\frac{du^{(2)}(x,0)}{dx}=U^{(2)}(x)=\frac{1}{\pi A}\int_{-\infty}^{\infty}\frac{\tau(s)ds}{s-x},\quad -\infty<x<\infty, \tag{1.3}$$

where

$$U^{(2)}(x)=\theta(-x-a)\frac{du^{(2)}}{dx}+\left[\theta(x+a)-\theta(x-a)\right]\frac{du^{(2)}}{dx}+\theta(x-a)\frac{du^{(2)}}{dx}, \tag{1.4}$$

$A=E/2(1-\nu^2)=2G(1-\chi^2)$, $\chi^2=(1-2\nu)/2(1-\nu)$, G is the shear modulus of material half-plane, $u^{(2)}(x,0)$ are the horizontal displacements of the boundary points of elastic half-plane.

Now, assuming that each differential element of the glue layer is subject to the condition of pure shear, we obtain the following condition [1–5]:

$$u^{(1)}(x)-u^{(2)}(x,0)=k\tau(x),\quad -\infty<x<\infty,$$

which when using the generalized functions may be presented as

$$U^{(1)}(x)=U^{(2)}(x)+k\tau'(x), \tag{1.5}$$

where $\tau'(x)=[\tau_a^+(x)+\tau_a^-(x)+\tau_0(x)]'$, $k=h_k/G_k$, $G_k=E_k/2(1+\nu_k)$ and character sroke implies the first-order derivative.

Below the following notations will be used for the Fourier transforms of $f(x)$ function:

$$\overline{f}(\sigma)=F\left[f(x)\right]=\int_{-\infty}^{\infty}f(x)e^{i\sigma x}dx,\quad f(x)=F^{-1}[\overline{f}(\sigma)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\sigma)\,e^{-i\sigma x}d\sigma,$$

$F[\cdot]$ is the operator of Fourier transform.

Now, after application of generalized Fourier transforms to (1.1), (1.3) and (1.5), the following functional equation on the real axis is obtained:

$$\left(\lambda_1^2+2\beta|\sigma|+\sigma^2\right)\bar{\tau}_\alpha^+(\sigma)+\left(\lambda_2^2+2\beta|\sigma|+\sigma^2\right)\bar{\tau}_0(\sigma)+ +\left(\lambda_3^2+2\beta|\sigma|+\sigma^2\right)\bar{\tau}_\alpha^-(\sigma)=\bar{f}_\beta(\sigma),\quad -\infty<\sigma<\infty, \tag{1.6}$$

where

$$\lambda_j^2=1/kE_jh\ (j=1,2,3),\ \beta=1/2kA,\ \bar{\tau}_\alpha^\pm(\sigma)=F[\tau_\alpha^\pm(x)],\ \bar{\tau}_0(\sigma)=F[\tau_0(x)], \bar{f}_\beta(\sigma)=P\left(\lambda_1^2e^{ib\sigma}+\lambda_3^2e^{-ib\sigma}\right)+\frac{u'_a}{k}e^{i\sigma a}+\frac{u'_{-a}}{k}e^{-i\sigma a}. \tag{1.7}$$

The equation (1.6) can be written in two forms:

$$(\lambda_3^2+2\beta|\sigma|+\sigma^2)\bar{\tau}(\sigma)+(\lambda_2^2-\lambda_3^2)\bar{\tau}_0(\sigma)+(\lambda_1^2-\lambda_3^2)\bar{\tau}_\alpha^+(\sigma)=\bar{f}_\beta(\sigma), \tag{1.8}$$

$$(\lambda_1^2+2\beta|\sigma|+\sigma^2)\bar{\tau}(\sigma)+(\lambda_2^2-\lambda_1^2)\bar{\tau}_0(\sigma)+(\lambda_3^2-\lambda_1^2)\bar{\tau}_\alpha^-(\sigma)=\bar{f}_\beta(\sigma). \tag{1.9}$$

Thus, the problem is reduced to the solution of functional equations in (1.8), (1.9) form. First, let us consider Eq.(1.8).

2. Solution of Functional Eq. (1.8) and Its Investigation. With that aim let us represent Eq.(1.8) as follows

$$\bar{\tau}(\sigma)+\frac{(\lambda_2^2-\lambda_3^2)\bar{\tau}_0(\sigma)}{\lambda_3^2+2\beta|\sigma|+\sigma^2}+\frac{(\lambda_1^2-\lambda_3^2)\bar{\tau}_\alpha^+(\sigma)}{\lambda_3^2+2\beta|\sigma|+\sigma^2}=\bar{g}_{\beta_3}(\sigma),\ -\infty<\sigma<\infty, \tag{2.1}$$

After applying the inverse Fourier transform to (2.1), we obtain

$$\tau(x)+(\lambda_2^2-\lambda_3^2)\int_{-a}^{a}K_{\beta_3}(x-s)\tau_0(s)ds+(\lambda_1^2-\lambda_3^2)\int_{-\infty}^{\infty}K_{\beta_3}(x-s)\tau_a^+(s)ds=g_{\beta_3}(x),\ -\infty<x<\infty, \tag{2.2}$$

where

$$\tau(x)=F^{-1}[\bar{\tau}(\sigma)],\ K_{\beta_3}(x)=F^{-1}[\bar{K}_{\beta_3}(\sigma)],\bar{K}_{\beta_3}(\sigma)=1/(\lambda_3^2+2\beta|\sigma|+\sigma^2), g_{\beta_3}(x)=F^{-1}[\bar{g}_{\beta_3}(\sigma)]=P[\lambda_1^2K_{\beta_3}(x-b)+\lambda_3^2K_{\beta_3}(x+b)]+\frac{u'_a}{k}K_{\beta_3}(x-a)+\frac{u'_{-a}}{k}K_{\beta_3}(x+a). \tag{2.3}$$

Now, when $|x|<a,\ \tau(x)=\tau_0(x)$ and when $x>a,\ \tau(x)=\tau_a^+(x)$, using (2.2) we obtain

$$\tau_0(x)+(\lambda_2^2-\lambda_3^2)\int_{-a}^{a}K_{\beta_3}(x-s)\tau_0(s)ds+(\lambda_1^2-\lambda_3^2)\int_{a}^{\infty}K_{\beta_3}(x-s)\tau(s)ds=g_{\beta_3}(x),\ |x|<a, \tau(x)+(\lambda_2^2-\lambda_3^2)\int_{-a}^{a}K_{\beta_3}(x-s)\tau_0(s)ds+(\lambda_1^2-\lambda_3^2)\int_{a}^{\infty}K_{\beta_3}(x-s)\tau(s)ds=g_{\beta_3}(x),\ a<x<\infty, \tag{2.4}$$

and now we can rewrite (2.2) in the form

$$\tau(x)=(\lambda_3^2-\lambda_2^2)\int_{-a}^{a}K_{\beta_3}(x-s)\tau_0(s)ds+(\lambda_3^2-\lambda_1^2)\int_{a}^{\infty}K_{\beta_3}(x-s)\tau(s)ds+g_{\beta_3}(x),\ -\infty<x<\infty. \tag{2.5}$$

Observing Eq.(1.9), it is worth-while to accept notations $K_{\beta_1}(x)=F^{-1}[\bar{K}_{\beta_1}(\sigma)]$ and $g_{\beta_1}(x)=F^{-1}[\bar{g}_{\beta_1}(\sigma)]$ instead of $K_{\beta_3}(x)$ and $g_{\beta_3}(x)$ that corresponds to the change of parameter λ_3^2 by λ_1^2.

Thus, the problem is reduced to solution of systems of integral equations in (2.4) form. To investigate the system of resolving Eq. (2.4), one should observe that [3–5]:

$$K_{\beta_3}(x)=(\gamma_1/\pi)\ln(b_2/b_1)+\gamma_1R(x), \tag{2.6}$$

where $b_1 = \dfrac{\lambda_3^2}{\beta + \sqrt{\beta^2 - \lambda_3^2}}$, $b_2 = \dfrac{\lambda_3^2}{\beta - \sqrt{\beta^2 - \lambda_3^2}}$, $\gamma_1 = \dfrac{1}{b_2 - b_1} = \dfrac{1}{2\sqrt{\beta^2 - \lambda_3^2}}$,

$R(x) = R_{b_1}(x) - R_{b_2}(x)$,

$$R_{b_j}(x) = -\frac{C}{\pi} + \sum_{k=1}^{\infty}(-1)^k \left[\frac{(b_j x)^{2k}}{\pi(2k)!}\left(\ln\frac{1}{|b_j x|} + 1 + \frac{1}{2} + \ldots + \frac{1}{2k} - C\right) - \frac{|b_j x|^{2k-1}}{2(2k-1)!}\right], \quad j = 1,2, \tag{2.7}$$

C is the Euler constant.

As it follows from (2.6), $K_{\beta_3}(x)$ is a continuous function in observation intervals and its value in $x = 0$ point is finite and equal to $K_{\beta_3}(0) = (\gamma_1/\pi)\ln(b_2/b_1)$.

Now, using Eq.(2.6), one can write the system of Eq. (2.4) in the following form:

$$\begin{aligned} \tau_0(x) &= \gamma_1(\lambda_3^2 - \lambda_2^2)\int_{-a}^{a} R(x-s)\tau_0(s)\,ds + (\lambda_3^2 - \lambda_1^2)\int_{a}^{\infty} K_{\beta_3}(x-s)\tau(s)\,ds + g_1(x), \ |x| < a, \\ \tau(x) &= (\lambda_3^2 - \lambda_2^2)\int_{-a}^{a} K_{\beta_3}(x-s)\tau_0(s)\,ds + (\lambda_3^2 - \lambda_1^2)\int_{a}^{\infty} K_{\beta_3}(x-s)\tau(s)\,ds + g_2(x), \ a < x < \infty, \end{aligned} \tag{2.8}$$

where $g_1(x) = g_{\beta_3}(x) + \dfrac{\gamma_1(\lambda_3^2 - \lambda_2^2)}{\pi}\ln(\dfrac{b_2}{b_1})\displaystyle\int_{-a}^{a}\tau_0(s)\,ds, \quad g_2(x) = g_{\beta_3}(x)$.

Then, taking into account (2.3) from (2.5) and (2.8), it is easy to see that $\tau(x)$ is a continuous function in the observation intervals and assumes finite values in $x = \pm a$, $x = \pm b$ points. It should be noted that such a behavior of $\tau(x)$ is conditioned on the presence of material interlayer of glue in the contact parts. In the absence of material interlayer in these points the overlay $\tau(x)$ has a logarithmic singularity [6]. After solving the system (2.8) one can find from (2.5) the values of contact stresses demanding $-\infty < x < -a$.

Now let us write the system of Eq. (2.8) in the operator form:

$$\varphi = K\varphi + g, \tag{2.9}$$

where

$$\begin{cases} \varphi = \begin{pmatrix} \tau_0 \\ \tau \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad K = \begin{pmatrix} \gamma_1(\lambda_3^2 - \lambda_2^2)k_{11} & (\lambda_3^2 - \lambda_1^2)k_{12} \\ (\lambda_3^2 - \lambda_2^2)k_{21} & (\lambda_3^2 - \lambda_1^2)k_{22} \end{pmatrix}, \\ k_{11}\tau_0 = \displaystyle\int_{-a}^{a} R(x-s)\tau_0(s)\,ds, \quad k_{12}\tau = \int_{a}^{\infty} K_{\beta_3}(x-s)\tau(s)\,ds, \\ k_{21}\tau_0 = \displaystyle\int_{-a}^{a} K_{\beta_3}(x-s)\tau_0(s)\,ds, \quad k_{22}\tau = \int_{a}^{\infty} K_{\beta_3}(x-s)\tau(s)\,ds. \end{cases} \tag{2.10}$$

Let consider Eq. (2.9) in B Banach space [7] by meanning the vector-function $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 \in L_1(-a, a)$, $x_2 \in L_1(a, \infty)$ and with norm

$$\|x\| = \max\left\{\|x_1\|_{L_1(-a,a)}, \|x_2\|_{L_1(a,\infty)}\right\}.$$

It is obvious that operator K acts in B space and is a Fredholm form operator. Hence, the operator Eq. (2.9) in B space may be solved by means of the method of successive approximations when $\| K \| < 1$, where

$$\|K\| = \max\left\{\left|\gamma_1\left(\lambda_3^2 - \lambda_2^2\right)\right|\|k_{11}\| + \left|\lambda_3^2 - \lambda_1^2\right|\|k_{12}\|,\ \left|\lambda_3^2 - \lambda_2^2\right|\|k_{21}\| + \left|\lambda_3^2 - \lambda_1^2\right|\|k_{22}\|\right\}.$$

Hence, we obtain the condition of system solution by means of the method of successive approximations:

$$\max\left\{|1-\lambda_2^2/\lambda_3^2| + \frac{1}{2}|1-\lambda_1^2/\lambda_3^2|,\ |1-\lambda_1^2/\lambda_3^2| + \frac{1}{2}|1-\lambda_2^2/\lambda_3^2|\right\} < 1. \qquad (2.11)$$

Similarly, from Eq. (1.9) we may obtain the system of integral equations in the form:

$$\begin{aligned}&\tau_0(x) + (\lambda_2^2-\lambda_1^2)\int_{-a}^{a} K_{\beta_1}(x-s)\tau_0(s)ds + (\lambda_3^2-\lambda_1^2)\int_{-\infty}^{-a} K_{\beta_1}(x-s)\tau(s)ds = g_{\beta_1}(x),\ |x|<a,\\ &\tau(x) + (\lambda_2^2-\lambda_1^2)\int_{-a}^{a} K_{\beta_1}(x-s)\tau_0(s)ds + (\lambda_3^2-\lambda_1^2)\int_{-\infty}^{-a} K_{\beta_1}(x-s)\tau(s)ds = g_{\beta 1}(x),\ -\infty<x<-a,\end{aligned} \qquad (2.12)$$

where for $-\infty < x < \infty$ $\tau(x)$ is already defined as follows:

$$\tau(x) = (\lambda_1^2 - \lambda_2^2)\int_{-a}^{a} K_{\beta_1}(x-s)\tau_0(s)ds + (\lambda_1^2 - \lambda_3^2)\int_{-\infty}^{-a} K_{\beta_1}(x-s)\tau(s)ds + g_{\beta_1}(x). \qquad (2.13)$$

By repeating to the system (2.12) above made reasonings, by analogous form we obtain the condition of the solving system by the method of successive approximations in the following form:

$$\max\left\{|1-\lambda_2^2/\lambda_1^2| + \frac{1}{2}|1-\lambda_3^2/\lambda_1^2|,\ |1-\lambda_3^2/\lambda_1^2| + \frac{1}{2}|1-\lambda_2^2/\lambda_1^2|\right\} < 1. \qquad (2.14)$$

It follows from both inequalities (2.11) and (2.14) that in the range of large variations of characteristic values of ratios λ_1^2/λ_3^2, λ_2^2/λ_3^2 and λ_3^2/λ_1^2, λ_2^2/λ_1^2 the problem in B space may be solved by means of the method of successive approximations.

In case of $\lambda_3^2 = \lambda_1^2$ the contact stresses are obtained by means of the method of successive approximations for all allowed values $0 < \lambda_2^2/\lambda_1^2 < 2$, and in case of $a \to 0$ for all values $0 < \lambda_j^2 < \infty$, $j = 1,3$.

For obtaining those inequalities the allowance was made for the fact that $\int_0^\infty K_{\beta_j}(x)\,dx = 1/2\lambda_j^2$, $K_{\beta j}(x) > 0$, $j = 1,3$, $0 < x < \infty$, since

$\int_0^\infty K_{\beta_3}(x)\,dx = \left(1/\pi(b_2 - b_1)\right)\int_0^\infty [K_1(b_1x) - K_2(b_2x)]\,dx$, $0 < b_2 \le 2b_1$, where

$\int_0^\infty K_i(b_ix)dx = \dfrac{\pi}{2b_i}$, $K_i(b_ix) = -\left[\cos(b_ix)\mathrm{Ci}(b_ix) + \sin(b_ix)\mathrm{si}(b_ix)\right]$, $i = 1,2$,

$\mathrm{si}(x)$ and $\mathrm{Ci}(x)$ are the sine integral and cosine integral respectively [8].

The values of $\tau(x)$ in the points $x = \pm a$ are obtained from (2.5) substituding $x = \pm a$, and the unknown coefficients u'_a and u'_{-a} can be found from conditions

$$u'_a = \frac{E_1 - E_2}{E_1 E_2 h}\left[\int_a^{\infty} \tau(s)ds - P\right], \quad u'_{-a} = \frac{E_2 - E_3}{E_2 E_3 h}\left[\int_{-\infty}^{-a} \tau(s)ds - P\right].$$

In case of homogeneous overlay, i.e. $\lambda_3^2 = \lambda_2^2 = \lambda_1^2$ (i.e., $E_3 = E_2 = E_1$), one has $u'_a = u'_{-a} = 0$, and from (2.2) we obtain for $\tau(x)$ a closed solution of corresponding problem in the Fourier integral form (see (2.3)):

$$\tau(x) = P\lambda_1^2\left[K_{\beta_1}(x-b) + K_{\beta_1}(x+b)\right],$$

or when the horizontal force P is applied in $x = 0$ point of overlay, the solution of problem is obtained in the form

$$\tau(x) = P\lambda_1^2 K_{\beta_1}(x) = \frac{P\lambda_1^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\sigma x} d\sigma}{\lambda_1^2 + 2\beta|\sigma| + \sigma^2}, \quad -\infty < x < \infty, \qquad (2.15)$$

From (2.15) without regard for the material interlayer, i.e. $k = 0$, $k = h_k / G_k$, the solution of the famous Melane problem (see [9]) is obtained in the form of expression $\tau(x) = \frac{P\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\sigma x} d\sigma}{\lambda + |\sigma|}$, $-\infty < x < \infty$, where the notation $\lambda = A / E_1 h$ is introduced.

Then, due to the representation for function $\overline{K}_{\beta_3}(\sigma)$ when $|\sigma| \to 0$, we obtain from (2.1) $\tau(x) = F^{-1}[\overline{\tau}(\sigma)]$ function when $|x| \to \infty$ is of the order of $O(x^{-2})$.

Received 02.07.2012

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