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*Mathematics*

# SOME REMARKS ON PROPERTIES OF ELEMENTS FROM COMPLEX BANACH ALGEBRAS

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In this work some properties related to von Neumann effect of commutators of elements from complex Banach algebras are discussed, as well as their "convexity".

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**I.** Let $\tau$ be a local convex topology on a complex algebra $\mathcal{A}$ with a unit and algebraic norm. Further, suppose the multiplication is continuous with respect on each coordinate and that identical map $\{\mathcal{A}, \|\cdot\|\} \to \{\mathcal{A}, \tau\}$ is also continuous. Let $\{P\}$ be the system defining algebraic seminorms, which generate topology $\tau$ [1].

Recall, that element $a \in \{\mathcal{A}, \tau\}$ belongs to the class $Gr(\{\mathcal{A}, \tau\})$, if there exists an element $b \in \{\mathcal{A}, \tau\}$ such that for every seminorm $\{P\}$:

$$\max\left\{P\left(\exp(-\lambda b)\cdot\exp(\bar{\lambda}a)\right),\ P\left(\exp(-\bar{\lambda}a)\cdot\exp(\lambda b)\right)\right\} = o\left(|\lambda|^{\frac{1}{2}}\right) \tag{1.1}$$

when $|\lambda| \to \infty$, $\lambda \in \mathbb{C}$ (see [2]).

*Proposition 1.1.* If element $a \in Gr(\{\mathcal{A}, \tau\})$, then element $b \in \{\mathcal{A}, \tau\}$ from the definition of the class $Gr(\{\mathcal{A}, \tau\})$ is unique.

*Proof.* Suppose there exists another element $\tilde{b} \in \{\mathcal{A}, \tau\}$ such that

$$\max\left\{P\left(\exp(-\lambda\tilde{b})\cdot\exp(\bar{\lambda}a)\right),\ P\left(\exp(-\bar{\lambda}a)\cdot\exp(\lambda\tilde{b})\right)\right\} = o\left(|\lambda|^{\frac{1}{2}}\right)$$

when $|\lambda| \to \infty$, $\lambda \in \mathbb{C}$ for every $P$.

Consider $\{\mathcal{A}, \tau\}$ is valued entire function

$f(\lambda) = \exp(-\lambda b)\exp(\lambda\tilde{b})$.

Then, for each seminorm $P$:

$$P(f(\lambda)) = P\left(\exp(-\lambda b)\cdot\exp(\lambda\tilde{b})\right) = P\left(\exp(-\lambda b)\exp(\bar{\lambda}a)\cdot\exp(-\bar{\lambda}a)\exp(\lambda\tilde{b})\right) \leqslant$$

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$$\leqslant P\left(\exp(-\lambda b)\exp(\bar{\lambda} a)\right)\cdot P\left(\exp(-\bar{\lambda} a)\exp(\lambda \tilde{b})\right)=o(|\lambda|),$$

when $|\lambda|\to\infty$, $\lambda\in\mathbb{C}$.

Applying (1.1) and Liouville's Theorem, we get $f(\lambda)\equiv f(0)=1$, from which it follows $\exp(\lambda b)=\exp(\lambda\tilde{b})$ and, therefore, $b=\tilde{b}$.

The Proposition 1.1 is proved.

*Proposition 1.2.* If $a_1,a_2\in Gr(\{\mathcal{A},\tau\})$ and $[a_1,b_2]=0$ or $[a_2,b_1]=0$, then $[a_1,a_2]=0$, $[b_1,b_2]=0$.

*Proof.* Indeed, since

$$f(\lambda)=\exp(-\lambda b_1)b_2\exp(\lambda b_1)=\exp(-\lambda b_1)\exp\left(\bar{\lambda}a_1\right)b_2\exp\left(\bar{\lambda}b_1\right)-$$
$$-\exp(-\lambda b_1)\left(\exp\left(\bar{\lambda}a_1\right)b_2-b_2\exp\left(\bar{\lambda}a_1\right)\right)\exp\left(-\bar{\lambda}a_1\right)\exp\left(\bar{\lambda}b_1\right)$$

and

$$g(\lambda)=\exp(-\lambda b_2)b_1\exp(\lambda b_2)=\exp(-\lambda b_2)\exp\left(\bar{\lambda}a_2\right)b_1\exp\left(-\bar{\lambda}a_2\right)\exp(\lambda b_2)-$$
$$-\exp(-\lambda b_2)\left(\exp\left(\bar{\lambda}a_2\right)b_1-b_1\exp\left(\bar{\lambda}a_2\right)\right)\exp\left(-\bar{\lambda}a_2\right)\exp\left(\bar{\lambda}b_2\right),$$

then from (1.1) and the condition $[a_1,b_2]=0$ or $[a_2,b_1]=0$, we get that $[b_1,b_2]=0$, changing $\lambda\to\mu=\bar{\lambda}$ gives $[a_1,a_2]=0$.

The Proposition 1.2 is proved.

As was proved in [2], the following generalization of von Neumann's Theorem holds for the elements from the class $Gr(\{\mathcal{A},\tau\})$ (see [1, 3]).

***Theorem 1.1.*** [2]. Let $a\in Gr(\{\mathcal{A},\tau\})$. Then, for every neighbourhood of zero $U\subset\{\mathcal{A},\tau\}$, there exists a neighbourhood of zero $V\subset\{\mathcal{A},\tau\}$ such, that if $x\in\mathcal{A}$, $\|x\|\leqslant 1$ and $[a,x]\in V$ , then $[b,x]\in U$.

It is clear, that all normal and quasinormal elements belong to the class $Gr(\{\mathcal{A},\tau\})$. For simplicity, in case of complex Banach algebra $\mathcal{A}$, let give an example of $a\in Gr(\mathcal{A})$ for which $[a,b]\neq 0$.

Fix $\sigma>0$ and $\alpha\in\mathbb{R}$. Denote by $B_\sigma(\alpha)$ the Banach algebra of all entire functions $f$ of the exponential type $\leqslant\sigma$, for which

$$\|f\|:=\sup_{\mathbb{R}}=\frac{|f(\lambda)|}{(1+|\lambda|)^\alpha}<\infty. \tag{1.2}$$

If $\alpha=0$, then $B_\sigma(\alpha)$ will be the classical Bernshtein space. If $\alpha\leqslant\beta$, then $B_\sigma(\alpha)\subset B_\sigma(\beta)$ [3]. Using Fragmente-Lindelof Theorem, we get

$$|f(\lambda)|\leqslant C_f(1+|\lambda|)^\alpha e^{\sigma|\mathrm{Im}\lambda|}. \tag{1.3}$$

From Cauchy's Theorem and inequalities (1.2) and (1.3) it follows that the operator $\delta=\frac{1}{i}\cdot\frac{d}{d\lambda}$ is acting continuously in space $B_\sigma(\alpha)$ and

$$\|\exp(it\delta)\|\sim(1+|t|)^{|\alpha|},\qquad t\to\pm\infty.$$

Consider the algebra $\mathcal{B}=M_2(BL(B_\sigma(\alpha)))$. With operator $\delta\in BL(B_\sigma(\alpha))$ and elements $a,b\in\mathcal{B}$, defined as follows:

$$a=\begin{pmatrix}\delta(1+i), & \delta(1+i)\\ 0, & \delta(1+i)\end{pmatrix},\qquad b=\begin{pmatrix}\delta(1-i), & 0\\ \delta(1-i), & \delta(1-i)\end{pmatrix}.$$

Then, it is easy to see that

$$[a,b]=2\delta^2\begin{pmatrix}\mathbf{1}, & 0\\ 0, & -\mathbf{1}\end{pmatrix}\neq 0.$$

Consider $\exp(-\lambda b)\cdot\exp(\bar{\lambda}a)$. We have

$$\exp(-\lambda b)\cdot\exp(\bar{\lambda}a)=\begin{bmatrix} e^{\delta\left(\bar{\lambda}(1+i)-\lambda(1-i)\right)}, & e^{\delta\left(\bar{\lambda}(1+i)-\lambda(1-i)\right)}\\ e^{\delta\left(\bar{\lambda}(1+i)-\lambda(1-i)\right)}, & e^{2\delta\left(\bar{\lambda}(1+i)-\lambda(1-i)\right)}\end{bmatrix},$$

similarly,

$$\exp(-\lambda a)\cdot\exp(\bar{\lambda}b)=\begin{bmatrix} e^{2\delta\left(\lambda(1-i)-\bar{\lambda}(1+i)\right)}, & e^{\delta\left(\lambda(1-i)-\bar{\lambda}(1+i)\right)}\\ e^{\delta\left(\lambda(1-i)-\bar{\lambda}(1+i)\right)}, & e^{\delta\left(\lambda(1-i)-\bar{\lambda}(1+i)\right)}\end{bmatrix}.$$

Let $\lambda=s+it\in\mathbb{C}$, then $\lambda(1-i)-\bar{\lambda}(1+i)=2i(t-s)$ and $\bar{\lambda}(1+i)-\lambda(1-i)=$ $=2i(s-t)$. Therefore,

$$\|\exp(-\lambda b)\cdot\exp(\bar{\lambda}a)\|=\left\|\begin{bmatrix} e^{2i(s-t)\delta}, & e^{2i(s-t)\delta}\\ e^{2i(s-t)\delta}, & e^{4i(s-t)\delta}\end{bmatrix}\right\|\sim(1+|\lambda|)^{4|\alpha|},$$

similarly,

$$\|\exp(-\bar{\lambda}a)\cdot\exp(\lambda b)\|\sim(1+|\lambda|)^{4|\alpha|}.$$

We will get the desired example, if we take $\alpha\in\mathbb{R}$ such that $|\alpha|<\dfrac{1}{8}$. For operator algebras the following theorem holds:

***Theorem 1.2.*** Let $X$ and $Y$ are complex Banach algebras, $A\in Gr(BL(X))$, $B\in Gr(BL(Y))$. Then for every $\varepsilon>0$ there exists $\delta>0$, such that if $T\in BL(X,Y)$, $\|T\|\leqslant R$ and $\|AT-TB\|<\delta$, then $\|A^{\oplus}T-TB^{\oplus}\|<\varepsilon$.

Note, that operators $A^{\oplus}\in BL(X)$ and $B^{\oplus}\in BL(Y)$ are conjugates of operators $A\in BL(X)$ and $B\in BL(Y)$ in the sense of class $Gr(\cdot)$ defined by (1.1).

The Theorem 1.2 is proved by the same scheme as the Theorem 1 from [2]. Consider the operator-valued entire function

$$F(\lambda)=\exp(-\lambda A^{\oplus})T\exp(\lambda B^{\oplus})=\exp(-\lambda A^{\oplus})\exp(\bar{\lambda}A)T\exp(-\bar{\lambda}B)\exp(\lambda B^{\oplus}))-$$
$$-\exp(-\lambda A^{\oplus})(\exp(\bar{\lambda}A)T-T\exp(\bar{\lambda}B))\exp(-\bar{\lambda}B)\exp(\lambda B^{\oplus}).$$

By Cauchy's integral formula

$$\|F'(0)\|\leqslant\frac{o(r)}{r}+\eta e^{4r}.$$

From here and the condition $\|AT-TB\|<\delta$ it follows that

$$\|A^{\oplus}T-TB^{\oplus}\|<\varepsilon.$$

The Theorem is proved.

The Theorem 1.2 has analogues corresponding to standard operators topologies, which satisfy the conditions of topology $\tau$. Note, that by the definition of the class $SGr(\mathcal{A})$ in [4], the element $b\in\mathcal{B}$ from the definition of $SGr(\mathcal{A})$ is unique as it is in Proposition 1.1

**II.** Let $\mathcal{A}$ be a complex Banach algebra with the unit $\mathbf{1}$, and let $\hat{a} = (a_k)_{k=0}^{\infty} \subset \mathcal{A}$ be a sequence of elements such that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\|a_k\|} = \rho(\hat{a}) < \infty. \qquad (2.1)$$

From (2.1) it follows that $\mathcal{A}$-valued function

$$E(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

is an entire $\mathcal{A}$-valued function of exponential type $\rho(\hat{a})$. The number $\rho(\hat{a})$ is called *generalized* spectral radius of the sequence $(\hat{a})$, which is the radius of the smallest circle centered at the origin and outside of which the following $\mathcal{A}$-valued power series converge

$$e(\lambda) = \sum_{k=0}^{\infty} \frac{a_k}{\lambda^{k+1}}\,.$$

For $\mathcal{A}$-valued function $E(z)$ the convex hull of the set of singularities is called conjugate diagram of the function $E(z)$, which we denote by $\mathcal{D}(\hat{a})$. Actually this set is the smallest convex set, which contains all singularities of $\mathcal{A}$-valued function $e(\lambda)$ [4].

Recall (see [4–6]), that a functional $\varphi \in \mathcal{A}^*$ is called to be a state, if $\|\varphi\| = \varphi(\mathbf{1}) = 1$. The set $St(\mathcal{A})$ of all states is a $\sigma(\mathcal{A}^*, \mathcal{A})$ compact convex subset of the conjugate space $\mathcal{A}^*$. In contrast to the set of complex homomorphisms, which can be empty in case of non-commutative algebras, the set of states is always non-empty. Recall, that for each element $a \in \mathcal{A}$ the set $V(a) = \{\varphi(a) : \varphi \in St(\mathcal{A})\}$ is called (algebraic) numerical image of element $a \in \mathcal{A}$. Not hard to see that for each $a \in \mathcal{A}$, $sp(a) \subset V(a)$, where $sp(a)$ is a spectrum of the element $a$. Using Han-Banach's Theorem, it can be proved that $\mathcal{A}$-valued function $E(z)$ and $e(\lambda)$ are related with integral representation:

$$E(z) = \frac{1}{2\pi i} \int_{\gamma} e(\lambda) e^{\lambda z} d\lambda\,,$$

where $\gamma$ is a closed contour containing $\mathcal{D}(\hat{a})$. If $K(\theta)$ is the support function of the set $\mathcal{D}(\hat{a})$ [7], then

$$\left\| E\left(re^{i\theta}\right) \right\| \leqslant L(\varepsilon) e^{[K(-\theta)+\varepsilon]r}$$

for every $\varepsilon > 0$, where $L(\varepsilon) = \dfrac{L}{2\pi} \max\limits_{\lambda \in \mathcal{D}_\varepsilon(\hat{a})} \|e(\lambda)\|$. Here $L$ is the length of the contour $\gamma$, and $\mathcal{D}_\varepsilon(\hat{a})$ ) is the $\varepsilon$-extension of $\mathcal{D}(\hat{a})$. For the ray $\ell(\theta_0) = \{z \in \mathbb{C} : \arg(z) = \theta_0\}$ the integral given by the formula

$$e(\lambda) = \int_{0}^{\infty e^{i\theta_0}} E(z) e^{-\lambda z} dz \qquad (2.2)$$

is an analytic $\mathcal{A}$-valued function in the half plane $\mathrm{R}e\left(\lambda e^{-i\theta_0}\right) > \rho(\hat{a}) + \delta$, where $\delta > 0$. Consider the indicator of growth $h(\theta)$ for the $\mathcal{A}$-valued function $E(z)$ (i.e.

$h(\theta)=\overline{\lim_{r\to\infty}}\frac{\ln\|E(re^{i\theta})\|}{r}$, $0\leqslant\theta\leqslant 2\pi$). Thus, the function $e(\lambda)$ is an $\mathcal{A}$-valued analytic function in the half-plane $\mathrm{R}e\left(\lambda e^{i\theta_0}\right)>h(\theta_0)$, for which the representation (2.2) holds, and, therefore, the $\mathcal{A}$-valued analogue of Polya's Theorem is hold too, $h(\theta)=K(-\theta)$.

Consider an important case when $a_k=a^k$, where $a\in\mathcal{A}$. In this case $E(z)=$ $=\exp(za)$ and $e(\lambda)=(\lambda\mathbf{1}-a)^{-1}$, $\rho(\hat{a})=\rho(a)$, where $\rho(a)$ is the spectral radius of the element $a\in\mathcal{A}$ and $\mathcal{D}(\hat{a})=\langle sp(a)\rangle$. Here $\langle sp(a)\rangle$ is the convex hull of the spectrum of the element $a\in\mathcal{A}$.

It is well known (see [8]), that

$$\begin{cases}\max\{\mathrm{R}e\lambda\,:\,\lambda\in sp(a)\}=\lim\limits_{r\to\infty}\dfrac{\ln\|\exp(ra)\|}{r}\,,\\[2ex] \max\{\mathrm{R}e\lambda\,:\,\lambda\in V(a)\}=\lim\limits_{t\to+0}\dfrac{\ln\|\exp(ta)\|}{t}\,.\end{cases}\tag{2.3}$$

Let $0\leqslant\theta\leqslant 2\pi$. Since $sp\left(ae^{-I\theta}\right)=e^{-i\theta}sp(a)$, assuming that $\mu=\lambda e^{-i\theta}$, where $\lambda\in sp(a)$, we get

$$\max\left\{\mathrm{R}e\mu:\mu\in sp\left(ae^{-i\theta}\right)\right\}=\max\left\{\mathrm{R}e\mu:\mu\in e^{-i\theta}sp(a)\right\}=$$
$$=\max\left\{\mathrm{R}e\left(\lambda e^{-i\theta}\right):\lambda\in sp(a)\right\}=K_{\langle sp(a)\rangle}(\theta).$$

Similarly, since $V\left(e^{-i\theta}a\right)=e^{-i\theta}V(a)$, then

$$\max\left\{\mathrm{R}e\mu:\mu\in V\left(e^{-i\theta}a\right)\right\}=\max\left\{\mathrm{R}e\mu:\mu\in e^{-i\theta}V(a)\right\}=$$
$$=\max\left\{\mathrm{R}e\left(\lambda e^{-i\theta}\right):\lambda\in V(a)\right\}=K_{V(a)}(\theta).$$

Using the first formula from (2.3), we get

$$K_{\langle sp(a)\rangle}(\theta)=\lim_{r\to\infty}\frac{\ln\|\exp(re^{-i\theta}a)\|}{r}=\ln\rho\left(\exp\left(e^{-i\theta}a\right)\right).$$

Since $\langle sp(a)\rangle\subset V(a)$, then for every $\theta\in[0,2\pi]$, $K_{\langle sp(a)\rangle}(\theta)\leqslant K_{V(a)}(\theta)$.

Now consider conditions of "covnexity" of the element $a\in\mathcal{A}$, i.e. when $\langle sp(a)\rangle=V(a)$.

Note, that all normal and subnormal elements are "convexoids".

Thus, if we want element $a\in\mathcal{A}$ be a "convexoid", we need to hold the condition $K_{\langle sp(a)\rangle}(\theta)\geqslant K_{V(a)}(\theta)$ for each $\theta\in[0,2\pi]$.

*Proposition 2.1.* If element $a\in\mathcal{A}$ is such that

$$\min_{0\leqslant\theta\leqslant 2\pi}\left\{\ln\rho\left(\exp\left(e^{-i\theta}a\right)\right)-\lim_{t\to+0}\frac{\left\|\mathbf{1}+te^{-i\theta}a\right\|-1}{t}\right\}\geqslant 0\,,\tag{2.4}$$

then $\langle sp(a)\rangle=V(a)$.

*Proof.* Since

$$K_{\langle sp(a)\rangle}(\theta)-K_{V(a)}(\theta)=\ln\rho\left(\exp\left(e^{-i\theta}a\right)\right)-\lim_{t\to+0}\frac{\ln\left\|\exp\left(te^{-i\theta}a\right)\right\|}{t}$$

then taking into consideration that

$$\lim_{t\to+0}\frac{\ln\left\|\exp\left(te^{-i\theta}a\right)\right\|}{t}=\lim_{t\to+0}\frac{\left\|\mathbf{1}+te^{-i\theta}a\right\|-1}{t}$$

and $K_{\langle sp(a)\rangle}(\theta)\geqslant K_{V(a)}(\theta)$ for every $\theta\in[0;2\pi]$, we get $\langle sp(a)\rangle=V(a)$. Note, that if $t=\dfrac{1}{n}$, the condition (2.4) can be written as

$$\min_{0\leqslant\theta\leqslant2\pi}\left\{\ln\rho\left(\exp\left(e^{-i\theta}a\right)\right)-\lim_{n\to\infty}\ln\left\|\sqrt[n]{\exp\left(e^{-i\theta}a\right)}\right\|^{n}\right\}\geqslant 0.$$

The Proposition is proved.

Let $a\in Gr(\mathcal{A})$ and $\theta\in[0;2\pi]$. Then by virtue of the definition class $Gr(\mathcal{A})$, on the ray $\ell(\theta)=\left\{\lambda=te^{-i\theta}\in\mathbb{C}:\arg(\lambda)=\theta\right\}$, we have

$$\begin{aligned}K_{\langle sp(a)\rangle}(\theta)&=\lim_{t\to\infty}\frac{\ln\left\|\exp\left(te^{-i\theta}a\right)\right\|}{t}=\\&=\lim_{t\to+\infty}\frac{\ln\left\|\exp\left(te^{i\theta}b\right)\cdot\exp\left(-te^{i\theta}b\right)\cdot\exp\left(te^{-i\theta}a\right)\right\|}{t}\leqslant\\&\leqslant\lim_{t\to+\infty}\left[\frac{\ln\left\|\exp\left(te^{i\theta}b\right)\right\|}{t}+\frac{\ln\left\|\exp\left(-te^{i\theta}b\right)\exp\left(te^{-i\theta}a\right)\right\|}{t}\right]=\\&=\lim_{t\to+\infty}\left[\frac{\ln\left\|\exp\left(te^{i\theta}b\right)\right\|}{t}+\frac{\ln o\left(t^{\frac{1}{2}}\right)}{t}\right]=\\&=\ln\rho\left(\exp\left(e^{i\theta}b\right)\right)=K_{\langle sp(b)\rangle}(-\theta).\end{aligned}$$

Similar reasoning shows that

$$K_{\langle sp(b)\rangle}(-\theta)=\ln\rho\left(\exp\left(e^{i\theta}b\right)\right)\leqslant\ln\rho\left(\exp\left(e^{-i\theta}a\right)\right)=K_{\langle sp(a)\rangle}(\theta).$$

Thus, for every $\theta\in[0;2\pi]$

$$K_{\langle sp(a)\rangle}(\theta)=K_{\langle sp(b)\rangle}(-\theta). \tag{2.5}$$

Taking into consideration (2.5), we get

$$K_{\langle sp(b)\rangle}(\theta)=K_{\langle sp(a)\rangle}(-\theta)=K_{\overline{\langle sp(a)\rangle}}(\theta).$$

Therefore, $\overline{\langle sp(a)\rangle}=\langle sp(b)\rangle$ and $\rho(a)=\rho(b)$. From (2.5) we get that for the indicators of growth $\langle sp\,(a)\rangle$ and $\langle sp\,(b)\rangle$, $h_{\langle sp\,(b)\rangle}(\theta)=K_{\langle sp\,(a)\rangle}(-\theta)$ by Polya's Theorem. Therefore, $h_{\langle sp\,(b)\rangle}(\theta)=h_{\langle sp\,(a)\rangle}(-\theta)$.

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