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Mathematics

A FUNCTIONAL REPRESENTATION OF FREE DE MORGAN ALGEBRAS

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It is well known that free Boolean algebra on n free generators is isomorphic to the Boolean algebra of Boolean functions of n variables. The free distributive lattice on n free generators is isomorphic to the lattice of monotone Boolean functions of n variables. In this paper we introduce the concept of De Morgan function and prove that the free De Morgan algebra on n free generators is isomorphic to the De Morgan algebra of De Morgan functions of n variables.

Keywords: Antichain, monotone Boolean function, De Morgan function, free De Morgan algebra.

1. Introduction. An algebra $Q(+,\cdot,\bar{\ },0,1)$ with two binary, one unary and two nullary operations is called a De Morgan algebra, if $Q(+,\cdot,0,1)$ is a bounded distributive lattice and $Q(+,\cdot,\bar{\ })$ satisfies the following identities [1–8]:

$$\overline{x+y}=\bar{x}\cdot\bar{y},\ \bar{\bar{x}}=x,\ \text{ where }\ \bar{\bar{x}}=\overline{(\bar{x})}.$$

The standard fuzzy algebra $F=([0,1];\ \max(x,y),\ \min(x,y),\ 1-x,0,1)$ is an example of De Morgan algebra.

Let us consider the following De Morgan algebras: $\mathbf{2}=(\{0,1\};\ \{+,\cdot,\bar{\ },0,1\})$; $\mathbf{3}=(\{0,a,1\};\{+,\cdot,\bar{\ },0,1\})$ with $\bar{a}=a$ and $\mathbf{4}=(\{0,a,b,1\};\{+,\cdot,\bar{\ },0,1\})$ with $\bar{a}=a$, $\bar{b}=b$, $a+b=1$, $a\cdot b=0$ (here 0 and 1 are the smallest and largest elements of distributive lattice and, so, the other values of operations are defined uniquely). The following result makes clear our approach for the definition of the concept of De Morgan functions.

Theorem 1.1 [9]. Every non-trivial subdirectly irreducible De Morgan algebra is isomorphic to one of the following algebras: **2, 3, 4**, where **2** is the unique non-trivial subdirectly irreducible Boolean algebra. □

Corollary 1.1 [10]. The free n-generated De Morgan algebra is a subalgebra of $\mathbf{4}^{4^n}$. □

2. De Morgan Functions. Let $2=B(+,\cdot,\bar{\ },0,1)$ be the two element Boolean algebra, where $B=\{0,1\}$. Denote $D=B\times B=\{(0,0),(1,0),(0,1),(1,1)\}=\{0,a,b,1\}$, where $0=(0,0)$, $a=(1,0)$, $b=(0,1)$, $1=(1,1)$. Defining $0+x=x$ and $1\cdot x=x$ for all $x\in D$ and $a+b=1$, $ab=0$, $\bar{0}=1$, $\bar{1}=0$, $\bar{a}=a$, $\bar{b}=b$, we get the De Morgan

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algebra **4** = $D(+,\cdot,\overline{}, 0,1)$. Note that $\overline{(u,v)} = (\overline{v},\overline{u})$, $(u_1,v_1)+(u_2,v_2)=(u_1+u_2, v_1+v_2)$ and $(u_1,v_1)\cdot(u_2,v_2)=(u_1\cdot u_2, v_1\cdot v_2)$ (here the operations on the right hand side are the operations of the Boolean algebra **2**). For $x \in D$ let

$$x^* = \begin{cases} x, & \text{if } x = 0, 1; \\ a, & \text{if } x = b; \\ b, & \text{if } x = a. \end{cases}$$

Also for $c = (c_1,\ldots,c_n)$, $d = (d_1,\ldots,d_n) \in D^n$ we say that d is a permitted modification of c, if for some k $(1 \le k \le n)$ we have $d_i = c_i$ for all $1 \le i \le n$, $i \ne k$, and

$$d_k = \begin{cases} a, & \text{if } c_k = 0; \\ 1, & \text{if } c_k = b. \end{cases}$$

Definition 2.1. A function $f : D^n \to D$ is called a De Morgan function, if the following conditions hold:

1) if $x_i \in \{0,1\}$, $i = \overline{1, n}$, then $f(x_1,\ldots,x_n) \in \{0,1\}$;

2) if $x_i \in D$, $i = \overline{1, n}$, then $f(x_1^*,\ldots,x_n^*) = (f(x_1,\ldots,x_n))^*$;

3) if $x \in D^n$ with $f(x) \ne b$ and y is a permitted modification of x, then $f(y) \in \{f(x), a\}$.

It is easy to see that the first condition is a consequence of the second condition.

Note that it follows from the first condition that every De Morgan function is an extension of some Boolean function. Also note that the constant functions $f=1$ and $f=0$ are De Morgan functions, but the constant functions $f=a$ and $f=b$ are not. Another examples of De Morgan functions are $f(x) = x$, $g(x) = \overline{x}$, $h(x,y) = x \cdot y$, $q(x,y) = x + y$, where the operations on the right hand side are the operations of the algebra **4**.

For $x_i \in D$ we denote by (y_i, z_i) the pair from $B \times B$, which is equal to x_i.

Definition 2.2. The function $f : D_n \to D$ is called an $\mathcal{A}$-function, if there exists a Boolean function $\varphi : B^{2n} \to B$ such that

$$f(x_1,\ldots x_n) = (\varphi(y_1,\ldots,y_n,\overline{z}_1,\ldots,\overline{z}_n), \varphi(z_1,\ldots,z_n,\overline{y}_1,\ldots,\overline{y}_n)) \tag{2.1}$$

for all $x_1,\ldots, x_n \in D$.

Before proceeding to the free De Morgan algebras, we are proving the following auxiliary results.

Proposition 2.1. The function $f : D^n \to D$ is an $\mathcal{A}$-function, iff it satisfies conditions (1) and (2) of Definition 2.1.

For an $\mathcal{A}$-function $f : D^n \to D$ there exists a unique Boolean function $\varphi : B^{2n} \to B$, which satisfies (2.1). To emphasize that φ is the function corresponding to f, we denote it by φ_f.

Theorem 2.1. The function $f : D^n \to D$ is a De Morgan function, iff it is an $\mathcal{A}$-function and φ_f is a monotone Boolean function. □

Corollary 2.1. There are m_{2n} De Morgan functions of n variables, where m_k is the number of monotone Boolean functions of k variables.

3. Free *n*-Generated De Morgan Algebras. Denote the set of all De Morgan functions of n variables by $\mathcal{D}_n$. For the functions $f, g : D^n \to D$ define $f + g$, $f \cdot g$ and $\overline{f}$ in the standard way, i.e. $(f + g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $\overline{f}(x) = \overline{f(x)}$,

$x \in D^n$, where the operations on the right hand side are the operations of De Morgan algebra **4**. We claim that D_n is closed under those operations, i.e. if $f, g \in D_n$, then $f+g, f \cdot g, \overline{f} \in D_n$. We can verify it straightforwardly using the definition of De Morgan function. But it is easier to prove it using Theorem 2.1.

Thus, we get an algebra $\mathcal{D}_n = D_n(+,\cdot,\overline{\ \ },0,1)$, which is obviously a De Morgan algebra.

Let $a=(a_1,a_2),\ b=(b_1,b_2) \in 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$. We say that $a \subseteq b$, if $a_1 \subseteq b_1$ and $a_2 \subseteq b_2$. In this way we get a partially ordered set $2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}(\subseteq)$. For any antichain $S \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$ we define the De Morgan function $f_S : D^n \to D$ in the following way:

$$f_S(x_1,\dots,x_n) = \sum_{s=(s_1,s_2)\in S} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i \right).$$

Note that f_S does not depend on the order of the elements in the set S. We set $f_\varnothing = 0$ and $f_{\{(\varnothing,\varnothing)\}} = 1$.

Let us consider the following functions $\delta_n^i = x_i,\ i=\overline{1,n}$, as functions from D_n to D. Obviously, δ_n^i is a De Morgan function.

The following main results are proved.

Theorem 3.1. For any De Morgan function f of n variables there exists a unique antichain $S \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}}$ such that $f = f_S$. □

Theorem 3.2. (functional representation theorem). The algebra $\mathcal{D}_n$ is a free De Morgan algebra with the system of free generators $\Delta = \{\delta_n^1,\dots,\delta_n^n\}$. Hence, every free n-generated De Morgan algebra is isomorphic to the De Morgan algebra $\mathcal{D}_n$. □

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