PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 2, p. 3–8

Mathematics

A TRANSCENDENCE RESULT FOR THE EQUATION $Dy = aDx$

V. A. ASLANYAN *

Mathematical Institute, University of Oxford, UK

An analogue of the Lindemann–Weierstrass theorem in differential setting for the differential equation $Dy = aDx$ is proved, where a is a non-constant parameter.

MSC2010: Primary 12H05, Secondary 12H20.

***Keywords*:** abstract differential equation, Ax-Schanuel theorem, Lindemann–Weierstrass theorem.

Introduction. Schanuel has conjectured that for any $\mathbb{Q}$-linearly independent complex numbers $z_1,\ldots,z_n$, one has

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(z_1,\ldots,z_n,e^{z_1},\ldots,e^{z_n}) \geq n, \tag{1}$$

where tr. deg. stands for the transcendence degree of the field extension (see [1]).

It generalises many results and conjectures in transcendental number theory (in particular the Lindemann–Weierstrass theorem) and is widely open. Schanuel's conjecture is also closely related to the model theory of the complex exponential field $\mathbb{C}_{\exp} = (\mathbb{C};+,\cdot,\exp)$ ([2]).

In [3] Ax proved a differential analogue of Schanuel's conjecture.

Theorem 1 [3]. Let $(K;+,\cdot,D,0,1)$ be a differential field of characteristic zero with the field of constants C and let $x_i,y_i \in K,\ i=1,\ldots,n$, be such that $Dy_i = y_i Dx_i$. If $\operatorname{tr.deg.}_C C(\bar{x},\bar{y}) \leq n$, then $x_1,\ldots,x_n$ are linearly dependent over $\mathbb{Q}$ modulo C, i.e. a $\mathbb{Q}$-linear combination of $x_1,\ldots,x_n$ is in C.

This is now known as the Ax-Schanuel theorem.

In this paper we will use the ideas of Ax's proof of the above Theorem to establish a transcendence result for the solutions (in a differential field K) of the differential equation

$$Dy = aDx, \tag{2}$$

where $a \in K \setminus C$ is a non-constant parameter. We formulate our main result below.

* E-mail: vahagn.aslanyan@maths.ox.ac.uk

Denote

$$A := \{x \in C(a)^{\mathrm{alg}} : \text{ there is } y \in C(a)^{\mathrm{alg}} \text{ such that } Dy = aDx\},$$

where for a field F we denote its algebraic closure by F^{alg}.

Theorem 2. Let $(x_1,y_1),\dots,(x_n,y_n) \in K^2$ be solutions of the differential equation $Dy = aDx$ such that $x_i \in C(a)^{\mathrm{alg}}$. Then $y_1,\dots,y_n$ are algebraically dependent over $C(a)$ if and only if $x_1,\dots,x_n$ are linearly dependent over C modulo A, i.e. some C-linear combination of $x_1,\dots,x_n$ is in A.

Note that since we require x_i's to be algebraic over $C(a)$, Theorem 2 is not an analogue of the Ax-Schanuel theorem. However it can be seen as an analogue of the Lindemann–Weierstrass theorem (in differential setting) for the equation $Dy = aDx$. Recall that the Lindemann–Weierstrass theorem states that for $\mathbb{Q}$-linearly independent complex algebraic numbers $\alpha_1,\dots,\alpha_n$ the exponentials $e^{\alpha_1},\dots,e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$ (this obviously follows from Schanuel's conjecture).

We will give some preliminaries on differential fields and present main ideas of Ax's method in the next section. Later, in the paper, they will be used to prove Theorem 2.

Preliminaries on Differential Fields and Ax's Method. First we present basic definitions and facts about differential fields. For more details we refer the reader to [4–7].

We assume all rings that we deal with are commutative rings with identity and have characteristic zero. A *derivation* of a ring R is a map $D : R \to R$ such that $\forall x,y\ D(x+y) = Dx + Dy$ and $\forall x,y\ D(xy) = xDy + yDx$. A *differential ring* is a ring equipped with a derivation.

For a differential field K the kernel of the derivation, that is, the set $C_K := \{x \in K : Dx = 0\}$ is a relatively algebraically closed subfield of K. It is called the *field of constants* of K.

If R is a differential ring, then the ring of *differential polynomials* over R is a differential ring extension defined as $R\{X\} = R[X, D(X), D^2(X),\dots]$ with $D(D^n(X)) = D^{n+1}(X)$. Thus, differential polynomials are of the form $P(X, DX,\dots,D^nX)$, where $P(X_0,\dots,X_n) \in R[X_0,\dots,X_n]$ is an algebraic polynomial over R. We can also consider differential polynomials in several variables, which are defined analogously. If $f(X_1,X_2,\dots,X_n)$ is such a polynomial, then the equation $f = 0$ is a *differential equation* over the given differential ring R. A tuple $(a_1,\dots,a_n) \in R^n$ is a solution to that equation if $f(a_1,\dots,a_n) = 0$.

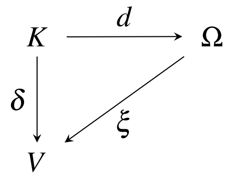
Now we present the main ideas of Ax's proof (see [3]), which will be used in the next section. We follow [8] and [9] in formulating and proving the facts below.

Let us start with the definition of a more general notion of derivation. Given a field K of characteristic 0 and a K-vector space V, a map $d : K \to V$ is called a *derivation*, if it is additive and satisfies Leibniz's rule, i.e.

$$d(x+y) = dx + dy,$$
$$d(xy) = xdy + ydx.$$

Let $C \subseteq K$ be a subfield.

Lemma 1. There is a K-vector space Ω and a derivation $d : K \to \Omega$ such that $C \subseteq \ker(d)$ and the following universal property holds: if V is a K-vector space and $\delta : K \to V$ is a derivation with $\delta|_C = 0$, then there is a unique K-linear map $\xi : \Omega \to V$ such that the following diagram commutes:



To see this we just need to take the abstract K-vector space generated by the set $\{dx : x \in K\}$ (where dx should be regarded just as a symbol for every $x \in K$) and quotient it out by relations

$$d(x+y) = dx + dy,\ d(xy) = xdy + ydx,\ dc = 0 \qquad (*)$$

for any $c \in C$. We define the map ξ in a natural way, i.e. $\xi(dx) = \delta(x)$ and then extend this to Ω by linearity. We need to check that this map is well-defined, i.e. it respects the above relations, which is straightforward.

The elements of Ω are called *abstract differential forms* on K over C (or *Kähler differentials*). The space Ω can be naturally embedded in the dual space of the vector space $\mathrm{Der}(K/C)$ of derivations of K over C. More precisely, if $\omega = dx \in \Omega$ is a differential form, then it can be considered as a linear functional $\omega : \mathrm{Der}(K/C) \to K$ defined by $\omega : \delta \mapsto \delta x$. One extends this definition to an arbitrary element of Ω using linearity. Thus differential forms can be thought of as linear forms on the vector space $\mathrm{Der}(K/C)$. If $\mathrm{tr.deg.}_C(K)$ is finite, then of course Ω is isomorphic to $\mathrm{Der}(K/C)^*$. In any case Ω^* is isomorphic to $\mathrm{Der}(K/C)$, which follows from the universal property of d.

If we want to emphasize that Ω is the vector space of differential forms on K over C, then we may write $\Omega_{K/C}$. But when it is clear from the context, which fields we refer to, we will omit the subscript K/C. Note that for a polynomial $P(X) \in C[X]$ one has $d(P(x)) = P'(x)dx$.

The importance of this universal vector space is that it allows us to translate algebraic dependence into linear dependence.

Remark. If $D : K \to K$ is a derivation, then it is easy to see that for a transcendental element t one can extend D to a derivation on $K(t)$ by defining $D(t)$ arbitrarily. Moreover, one can extend a derivation to an algebraic extension E. Indeed, if x is algebraic over K, then let $P \in K[X]$ be its minimal polynomial over K. The equation $P(x) = 0$ forces us to define Dx in a natural way. That is, Dx should be defined from the equality $P^D(x) + P'(x) \cdot Dx = 0$, where P^D is the polynomial obtained from P by differentiating its coefficients. Since we are in characteristic zero, $P'(X)$ is a non-zero polynomial with lower degree than P and, therefore, it does not vanish at x.

Lemma 2. Arbitrary elements $x_1,\ldots,x_n \in K$ are algebraically dependent over C if and only if $dx_1,\ldots,dx_n$ are linearly dependent over K.

Proof. Suppose $x_1,\ldots,x_n \in K$ are algebraically dependent over C. That is, there is a polynomial $P(X_1,\ldots,X_n) \in C[X_1,\ldots,X_n]$ such that $P(\bar{x}) = 0$. We can choose P to be of minimal degree with this property. Then $\sum \dfrac{\partial P}{\partial X_i}(\bar{x})dx_i = 0$. Since $\dfrac{\partial P}{\partial X_i}$ does not vanish at $\bar{x}$, $dx_1,\ldots,dx_n$ are linearly dependent over K.

Conversely let $x_1,\ldots,x_n$ be algebraically independent over C. By the Remark above, we can define derivations $D_i : K \to K$ such that D_i vanishes on C and $D_i(x_i) = 1,\ D_i(x_j) = 0$ for $i \neq j$. By the universal property of Ω, we find K-linear maps $\xi_i : \Omega \to K$ such that $\xi_i \circ d = D_i$.

If $f_i \in K$ with $\sum_i f_i dx_i = 0$, then applying ξ_i we see that $f_i = 0$. □

We conclude immediately that $\mathrm{l.dim}_K(\Omega) = \mathrm{tr.deg.}_C(K)$, where l.dim stands for the linear dimension of a vector space. In particular, an element $c \in K$ is algebraic over C if and only if $dc = 0$. Thus $\ker(d)$ is equal to the relative algebraic closure of C in K.

From now on we assume K is a differential field with a derivation D and constant field C. Let $\Omega = \Omega_{K/C}$ and $\xi : \Omega \to K$ be a linear map with $\xi(dx) = Dx$. In this case $\ker(d) = C$.

We define the *Lie derivative* $L_D : \Omega \to \Omega$ as follows:

$$L_D\left(\sum f_i dx_i\right) = \sum Df_i \cdot dx_i + f_i \cdot d(Dx_i).$$

It is easy to check that this map is well defined, i.e. it respects the relations $(*)$.

Also, one can easily verify the following properties of the Lie derivative L_D:

$$\begin{aligned} &L_D(\omega_1 + \omega_2) = L_D(\omega_1) + L_D(\omega_2) \ \text{ for } \omega_1, \omega_2 \in \Omega, \\ &L_D(f\omega) = D(f)\omega + fL_D(\omega) \ \text{ for } \omega \in \Omega,\ f \in K, \\ &L_D(dx) = d(Dx) \ \text{ for } x \in K. \end{aligned}$$

The following result allows us to translate linear dependence of vectors from Ω over K into linear dependence over C.

Lemma 3. Let $\omega_i \in \Omega$ be linearly dependent over K and $L_D(\omega_i) = 0$ for $i = 1,\ldots,n$. Then $\omega_1,\ldots,\omega_n$ are C-linearly dependent.

Proof. Suppose $s\sum_i f_i\omega_i = 0$ for some $f_i \in K$. Without loss of generality we may assume that this linear dependence is minimal, that is, a maximal number of $f_1,\ldots,f_n$ are zero. Dividing by a non-zero coefficient we may also assume that one of them, say f_1, is equal to 1.

Apply L_D to that linear combination. We have

$$0 = L_D\Big(\sum f_i\omega_i\Big) = \sum(D(f_i)\omega_i + f_iL_D(\omega_i)) = \sum D(f_i)\omega_i.$$

Since $D(f_1) = 0$, this new linear combination contains one more zero than the initial one. Hence, all $D(f_i)$'s must be equal to zero. This shows that f_i's are constants, and we are done. □

Proof of Theorem 2. In this section we will prove Theorem 2. But first let us make some observations, which will make the choice of the set A clear.

Let (x,y) be a non-constant solution to (2) with $\operatorname{tr.deg}_{\cdot C} C(x,y) \leq 1$. Then for some polynomial $P(X,Y) \in C[X,Y]$ we will have $P(x,y) = 0$. We may assume P is of minimal degree with this property. Differentiating this equality, we get

$$\frac{\partial P}{\partial X}(x,y) \cdot Dx + \frac{\partial P}{\partial Y}(x,y) \cdot Dy = 0.$$

Taking into account the equality $Dy = aDx$, we express a as a rational function of x and y (since $Dx \neq 0$). That is, $a \in C(x,y)^{\text{alg}} = C(x)^{\text{alg}} = C(y)^{\text{alg}}$. This implies $x,y \in C(a)^{\text{alg}}$. Thus, for just one solution (x,y), x and y will be algebraically dependent over C if and only if they both are in an algebraic dependence with a, again over C.

Therefore,

$$A = \{x \in C(a)^{\text{alg}} : \text{ there is } y \in C(a)^{\text{alg}} \text{ such that } Dy = aDx\} =$$
$$= \{x \in K : \text{there is } y \in K \text{ with } Dy = aDx \text{ and } \operatorname{tr.deg}_{\cdot C} C(x,y) \leq 1\}.$$

We can interpret this as follows. For a solution (x,y) we want the transcendence degree of x,y over C to be 2. Then A can be thought of as the set of all counterexamples to this, that is, the set of all x, for which the transcendence degree is less than expected.

Proof of Theorem 2. Suppose $y_1,\ldots,y_n$ are algebraically dependent over $C(a)$. We may assume that $x_i \notin C$ for each i. Let

$$V := \langle dx_1,\ldots,dx_n,dy_1,\ldots,dy_n\rangle = \langle da,dy_1,\ldots,dy_n\rangle$$

be the K-linear span of $dx_1,\ldots,dx_n,dy_1,\ldots,dy_n$ in Ω ($dx_1,\ldots,dx_n$ are all in the linear span of da since $x_i \in (C(a))^{\text{alg}}$). Then, by Lemma 2, $\operatorname{l.dim}_K V \leq n$. Consider the differential forms $\omega_i := dy_i - adx_i \in V$. The restriction of ξ to V is a linear map, whose image is K and hence has dimension 1 over K. Notice that $\xi(\omega_i) = Dy_i - aDx_i = 0$, i.e. each ω_i is in the kernel of ξ. The rank-nullity theorem now yields that $\operatorname{l.dim}_K(\ker(\xi)) = n-1$ and so $\omega_1,\ldots,\omega_n$ are linearly dependent over K. Now let us calculate $L_D(\omega_i)$. We have

$$\begin{aligned} L_D(\omega_i) &= d(Dy_i) - Da \cdot dx_i - ad(Dx_i) \\ &= d(aDx_i) - Da \cdot dx_i - ad(Dx_i) \\ &= a \cdot d(Dx_i) + Dx_i \cdot da - Da \cdot dx_i - a \cdot d(Dx_i) \\ &= Dx_i \cdot da - Da \cdot dx_i = 0. \end{aligned}$$

To get the last equality we use the following argument. Since a and x_i are algebraically dependent over C, da and dx_i are linearly dependent over K. This means that for some $f \in K$ we have $dx_i = f \cdot da$. Applying ξ we see that $Dx_i = f \cdot Da$, which implies $Dx_i \cdot da - Da \cdot dx_i = 0$.

Thus, we can apply Lemma 3, which implies that $\omega_1,\ldots,\omega_n$ are C-linearly dependent. Let $c_1,\ldots,c_n$ be constants such that $\sum\limits_i c_i\omega_i = 0$. This is equivalent to

$$d\left(\sum_i c_i y_i\right) = a \cdot d\left(\sum_i c_i x_i\right).$$

Denoting $x = \sum_i c_i x_i$ and $y = \sum_i c_i y_i$, we will have $dy = adx$. Applying ξ, we get $Dy = aDx$. That is, (x,y) is a solution to equation (2) and x is a C-linear combination of $x_1, \ldots, x_n$. Since dx and dy are linearly dependent, x and y are algebraically dependent over C, which yields $x \in A$. This is the desired result.

The converse is much simpler. Suppose $\sum_i c_i x_i \in A$ for some constants $c_1, \ldots, c_n$. Denote $x = \sum_i c_i x_i$ and $y = \sum_i c_i y_i$. The pair (x,y) is a solution to (2) and since $x \in A$, we conclude that $y \in C(a)^{\mathrm{alg}}$. This implies that $y_1, \ldots, y_n$ are algebraically dependent over $C(a)$. $\square$

Received 07.03.2016

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