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Mathematics

ON THE CONVERGENCE OF FOURIER–LAPLACE SERIES

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In the present paper we prove the following theorem. For any $\varepsilon > 0$ there exists a measurable set $G \subset S^3$ with measure $\operatorname{mes} G > 4\pi - \varepsilon$, such that for each $f(x) \in L^1(S^3)$ there is a function $g(x) \in L^1(S^3)$, coinciding with $f(x)$ on G with the following properties. Its Fourier–Laplace series converges to $g(x)$ in metrics $L^1(S^3)$ and the inequality holds $\sup_N \left\| \sum_{n=1}^{N} Y_n[g,(\theta,\varphi)] \right\|_{L^1(S^3)} << 3\|g\|_{L^1(S^3)} \leq 12 \| f \|$.

Keywords: spherical harmonics, Legendre polynomials, convergence of Fourier series.

Let S^3 be the unit sphere in three-dimensional Euclidean space R^3. The Cartesian and spherical coordinates of a point $x = (x_1, x_2, x_3) \in S^3$ are connected by relations

$$x_1 = \sin\theta\cos\varphi, \quad x_2 = \sin\theta\sin\varphi, \quad x_3 = \cos\theta, \ \theta \in [0,\pi], \ \varphi \in [0,2\pi].$$

Let $L^1(S^3)$ be the class of functions $f(x) = f(\theta,\varphi), \quad x \in S^3$, with bounded integral $\iint_{S^3} |f(x)| ds$, where $ds = \sin\theta d\theta d\varphi$ is the area element of the surface S^3. We denote by $\|f\|_{L^1}$ the norm of $f(x)$ in $L^1(S^3)$, i.e. $\|f\|_{L^1(S^3)} = \iint_{S^3} |f(x)| ds$. It is known (see [1–3]) that in R^3 there are $2n+1$ linearly independent spherical harmonics of order n:

$$\begin{gathered} p_n(\cos\theta), \ p_n^m(\cos\theta)\cos m\varphi, \ p_n^m(\cos\theta)\sin m\varphi, \\ m = 1,2,\dots,n, \ n = 0,1,2,\dots, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi, \end{gathered} \tag{1}$$

where $p_n(t)$ are standard, $p_n^m(t)$ are adjoint Legendre polynomials. Any spherical function $Y_n(\theta,\varphi)$ of order n may be represented as a combination of functions (1):

$$Y_n(\theta,\varphi) = \frac{1}{2}\alpha_{n,0} p_n(\cos\theta) + \sum_{m=1}^{n} p_n^m(\cos\theta)\left[\alpha_n^{(m)} \cos m\varphi + \beta_n^{(m)} \sin m\varphi\right].$$

Note that the functions $Y_n(\theta,\varphi)$ as well as the functions (1) form orthogonal systems on S^3 and

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$$\|p_n(\cos\theta)\|_{L^2(S^3)} = \frac{4\pi}{2n+1},$$
$$\|p_n^m(\cos\theta)\cos m\varphi\|^2_{L^2(S^3)} = \|p_n^m(\cos\theta)\sin m\varphi\|^2_{L^2(S^3)} = \frac{2\pi}{n+1}\cdot\frac{(n+m)!}{(n-m)!}. \tag{2}$$

Note that there is a renewed interest to spherical functions (see for instance [1–5]). For $f(x)\in L(S^3)$ we put

$$a_n^{(m)}(f) = \frac{2n+1}{2\pi}\cdot\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)p_n^m(\cos\theta)\cos m\varphi\sin\theta\, d\theta\, d\varphi,$$
$$b_n^{(m)}(f) = \frac{2n+1}{2\pi}\cdot\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)p_n^m(\cos\theta)\sin m\varphi\ \sin\theta\, d\theta\, d\varphi,$$
$$Y_n[f,(\theta,\varphi)] = \frac{1}{2}a_n^{(0)}(f)p_n(\cos\theta) + \tag{3}$$
$$+\sum_{m=1}^{n} p_n^{(m)}(\cos\theta)[a_n^{(m)}(f)\cos m\varphi + b_n^{(m)}(f)\sin m\varphi].$$

The series $\sum_{n=0}^{\infty} Y_n[f,(\theta,\varphi)]$ is called Fourier–Laplace series for the function $f(\theta,\varphi)$.

In 1975 A.Bonami and E.Clerc [6] proved that there is a function $f(\theta,\varphi)\in L^1(S^3)$ such that its Fourier–Laplace series diverges in the metrics $L^1(S^3)$:

$$\overline{\lim_{N\to\infty}}\left\|\sum_{n=1}^{N} Y_n\left[f,(\theta,\varphi)\right] - f(\theta,\varphi)\right\|_{L^1} = +\infty.$$

In the present work we prove the following theorem.

Theorem. For any $\varepsilon > 0$ there is a measurable set $G\subset S^3$ with measure $\mathrm{mes}G > 4\pi-\varepsilon$, such that for each $f(x)\in L^1(S^3)$ there is a function $g(x)\in L^1(S^3)$ coinciding with $f(x)$ on G, whereas its Fourier–Laplace series converges to $g(x)$ in $L^1(S^3)$ and

$$\sup_N\left\|\sum_{n=1}^{N} Y_n[g,(\theta,\varphi)]\right\|_{L^1(S^3)} < 3\|g\|_{L^1(S^3)} \le 12\|f\|.$$

Remark. This Theorem is a strengthening of M.G. Grigorian's result (Theorem 1, [5]). To prove the Theorem we use the method developed by M.G. Grigorian in [7, 8]. However the idea of improving the convergence of Fourier series through changing the function being decomposed on a small measure set, belongs to D.E. Menshoff [9, 10]. In this direction a number of interesting results were obtained in [11, 13]. We'll use the following Lemma to prove the Theorem.

Lemma. Given the function $f(\theta,\varphi)\in L^1(S^3)$, $0<\varepsilon<1$ and $N_0>1$. There are a measurable set $E\subset S^3$, a function $g(\theta,\varphi)\in L(S^3)$ and a polynomial by spherical harmonics having the form $P(\theta,\varphi) = \sum_{k=N_0}^{\tilde{N}} Y_k(\theta,\varphi)$, satisfying the following conditions:

1. $\mathrm{mes}E > 4\pi-\varepsilon$,

2. $g(\theta,\varphi)=f(\theta,\varphi)$ on E,
3. $\frac{1}{2}\|f\|_{L^1(S^3)}\le\|g\|_{L^1(S^3)}\le 4\|f\|_{L^1(S^3)}$,
4. $\|g-p\|_{L^2(S^3)}<\varepsilon$,
5. $\sup\limits_{N_0\le m\le\bar{N}}\left\|\sum\limits_{k=N_0}^{m}Y_k\right\|_{L^1(S^3)}\le 4\|f\|_{L^1(S^3)}$.

The proof of this Lemma is similar to the one used in [7] for Lemma 2.

Proof of Theorem. Let

$$f_1(\theta,\varphi),\ f_2(\theta,\varphi),\dots,\ f_n(\theta,\varphi),\dots \tag{4}$$

be a sequence of polynomials by spherical harmonics $Y_n(\theta,\varphi)$ with rational coefficients. Repeatedly applying the Lemma we get the sequences of functions $\{\bar{g}_n(\theta,\varphi)\}$, sets $\{\bar{E}_n\}$ and polynomials

$$\bar{p}_n(\theta,\varphi)=\sum_{k=M_{n-1}}^{M_n-1}Y_k(\theta,\varphi),\ \ M_n<M_{n+1}, \tag{5}$$

satisfying the conditions:

$$\begin{gathered}\bar{g}_n(\theta,\varphi)=f_n(\theta,\varphi)\ \text{ on }\ \bar{E}_n\subset S^3,\\ \left|\bar{E}_n\right|>4\pi-\frac{\varepsilon}{2^n},\\ \frac{1}{2}\|f_n\|_{L^1(S^3)}\le\|\bar{g}_n\|_{L^1(S^3)}\le 4\|f_n\|_{L^1(S^3)},\\ \|\bar{p}_n-\bar{g}_n\|_{L^2(S^3)}<2^{-4n},\\ \sup_{M_{n-1}\le m<M_n}\left\|\sum_{k=M_{n-1}}^{m}Y_k\right\|_{L^1(S^3)}\le 4\|f_n\|_{L^1(S^3)}.\end{gathered} \tag{6}$$

We put $E=\bigcap\limits_{n=1}^{\infty}\bar{E}_n$. Obviously (see condition 1 of Lemma), $\text{mes}E>4\pi-\varepsilon$. Let $f(x)=f(\theta,\varphi)\in L^1(S^3)$. It is easy to see that one can choose a subsequence $\left\{f_{k_n}(x)\right\}$ from the sequence (4), such that

$$\lim_{N\to\infty}\iint_{S^3}\left|\sum_{n=1}^{N}f_{k_n}(x)-f(x)\right|ds=0, \tag{7}$$

$$\iint_{S^3}\left|f_{k_n}(x)\right|ds<2^{-8n},\ n\ge 2. \tag{8}$$

Assume that the functions $g_1(x),\dots,g_{\upsilon-1}(x)$ and the polynomials $P_n(x)=\sum\limits_{k=m_n}^{\bar{m}_n}Y_k(x)$, $m_{n+1}>\bar{m}_n$, $n=1,2,\dots,\upsilon-1$, satisfying the conditions

$$g_n(x)=f_{k_n}(x),\ x\in E,\ \ n\le\upsilon-1, \tag{9}$$

$$\frac{1}{2}\left\|f_{k_n}\right\|_{L^1(S^3)}<\iint_{S^3}\left|g_n(x)\right|ds<2^{-2(n-1)}, \tag{10}$$

$$\iint_{S^3}\left|\sum_{i=1}^{n}\left[P_i(x)-g_i(x)\right]\right|^2 dx<2^{-2n}, \tag{11}$$

$$\max_{m_n\le i<\overline{m}_n}\iint_{S^3}\left|\sum_{k=m_n}^{i}Y_k(x)\right|ds<2^{-n}, \tag{12}$$

are already defined.

We take a function $f_{k_\upsilon}(x)$ from the sequence (4) such that

$$\iint_{S^3}\left|f_{k_n}(x)-\left\{f_{k_\upsilon}(x)-\sum_{i=1}^{\upsilon-1}\left[P_i(x)-g_i(x)\right]\right\}\right|ds<2^{-8\upsilon}. \tag{13}$$

Taking into account (8) and (11) , we have

$$\iint_{S^3}\left|f_{k_\upsilon}(x)-\sum_{i=1}^{\upsilon-1}\left[P_i(x)-g_i(x)\right]\right|ds<2\cdot 2^{-2(\upsilon-1)}. \tag{14}$$

From (13) it follows $\iint_{S^3}\left|f_{k_\upsilon}(x)\right|ds<2^{-\upsilon}$. We put

$$\begin{aligned} &g_\upsilon(x)=f_{k_\upsilon}(x)+\left[\overline{g}_{k_n}(x)-f_{k_n}(x)\right],\\ &P_\upsilon(x)=\overline{P}_{k_n}(x)=\sum_{k=m_\upsilon}^{\overline{m}_\upsilon}Y_k(x), \end{aligned} \tag{15}$$

where $m_\upsilon=M_{k_n-1}$, $\overline{m}_\upsilon=M_{k_n}-1$.

Taking into account (9), (10)–(13), (15), we get $g_\upsilon(x)=f_{k_\upsilon}(x),\ x\in E$,

$$\begin{aligned} \iint_{S^3}\left|g_\upsilon(x)\right|ds\le\iint_{S^3}\left|\sum_{i=1}^{\upsilon-1}\left[P_i(x)-g_i(x)\right]\right|ds+\iint_{S^3}|\overline{g}_{k_n}(x)|\,ds+\\ +\iint|f_{k_n}(x)-\left\{f_{k_\upsilon}(x)-\left[P_i(x)-g_i(x)\right]\right\}|\,ds<5\cdot 2^{-2(\upsilon-1)}, \end{aligned} \tag{16}$$

$$\begin{aligned} \left(\iint_{S^3}\left|\sum_{i=1}^{\upsilon}\left[P_i(x)-g_i(x)\right]\right|^2 ds\right)^{1/2}\le\left(\iint_{S^3}\left|f_{k_n}-\left\{f_{k_\upsilon}-\sum_{i=1}^{\upsilon-1}\left[P_i(x)-g_i(x)\right]\right\}\right|^2 ds\right)^{1/2}+\\ +\left(\iint_{S^3}\left|\overline{P}_{k_\upsilon}(x)-\overline{g}_{k_\upsilon}(x)\right|^2 ds\right)^{1/2}<2^{-2\upsilon}. \end{aligned} \tag{17}$$

It is clear that one can define by induction a sequence of functions $\{g_\upsilon(x)\}$ and polynomials $\{P_\upsilon(x)\}$, satisfying the conditions (16) and (17) for all $\upsilon\ge 1$. From (16) it follows

$$\iint_{S^3}\left|\sum_{i=1}^{\infty}g_i(x)\right|ds<\infty. \tag{18}$$

We define the function $g(x)$ and its series by spherical harmonics in the following way

$$g(x)=\sum_{i=1}^{\infty}g_i(x),\quad \sum_{\upsilon=1}^{\infty}P_\upsilon(x)=\sum_{\upsilon=1}^{\infty}\left(\sum_{k=m_\upsilon}^{\overline{m}_\upsilon}Y_k(x)\right)=\sum_{n=1}^{\infty}Y_n(x),$$

$$m_{\upsilon+1} > \overline{m}_\upsilon,\ Y_n(x) \equiv 0, \quad \text{for} \quad n \in \bigcup_{\upsilon=1}^{\infty} \left[\overline{m}_\upsilon + 1, m_{\upsilon+1}\right].$$

From (18) we have

$$g(x) \in L(S^3),\ \ g(x) = f(x) \quad \text{on} \quad E\ ,$$

$$\frac{1}{2}\|f\|_{L^1(S^3)} < \|g\|_{L^1(S^3)} \le 4\|f\|_{L^1(S^3)}\ .$$

Let m be an arbitrary natural number $m > m_{\upsilon_0}$. Then for some $\upsilon > \upsilon_0$ we have $m_\upsilon \le m \le m_{\upsilon+1}$, and from (10) and (15) we obtain

$$\iint_{S^3}\left|\sum_{k=1}^{m} Y_k(x) - g(x)\right| ds \le \iint_{S^3}\left|\sum_{i=1}^{\upsilon-1} P_i(x) - g_i(x)\right| ds +$$

$$\sum_{k=\upsilon}^{\infty}\iint_{S^3} |\, g_k(x)\,|\, ds + \iint_{S^3}\left|\sum_{k=m_\upsilon}^{m} |\, Y_k(x)\,|\, ds\right| < 2^{-\upsilon+3}.$$

$$\left\|\sum_{k=1}^{m} Y_k\right\|_{L^1} = \iint_{S^3}\left|\sum_{k=1}^{m} Y_k(x)\right| ds \le \sum_{\upsilon=1}^{\infty}\left(\max_{m_\upsilon \le m < \overline{m}} \iint_{S^3}\left|\sum_{k=m_\upsilon}^{m} Y_k(x)\right| ds\right) + 2\iint_{S^3}\left|g_1(x)\right| ds \le$$

$$\le 3\iint_{S^3}\left|g(x)\right| ds \le 12\iint_{S^3}\left|f(x)\right| ds.$$

The Theorem is thus proved.

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Ֆուրյե–Լապլասի շարքերի զուգամիտության մասին

Աշխատանքում ապացուցված է հետևյալ թեորեմը: Յանկացած դրական ε թվի համար գոյություն ունի $\mathrm{mes}G > 4\pi - \varepsilon$ չափով այնպիսի չափելի $G \subset S^3$ բազմություն, որ ցանկացած $f(x) \in L^1(S^3)$ ֆունկցիայի համար կարելի է գտնել $g(x) \in L^1(S^3)$ ֆունկցիա, որը համընկնում է $f(x)$-ի հետ G բազմության վրա և որի Ֆուրյե–Լապլասի շարքը զուգամիտում է $g(x)$ -ին $L^1(S^3)$ մետրիկայով և տեղի ունի հետևյալ անհավասարությունը. $\sup_N \left\| \sum_{n=1}^{N} Y_n[g,(\theta,\varphi)] \right\|_{L^1(S^3)} << 3\|g\|_{L^1(S^3)} \le 12 \| f \|$:

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О сходимости рядов Фурье–Лапласа

В статье доказывается следующая теорема. Пусть ε – любое положительное число. Тогда существует измеримое множество $G \subset S^3$ с мерой $\mathrm{mes}G > 4\pi - \varepsilon$ такое, что для каждой функции $f(x) \in L^1(S^3)$ можно найти функцию $g(x) \in L^1(S^3)$, совпадающую с $f(x)$ на множестве G, что ее ряд Фурье–Лапласа сходится к $g(x)$ в метрике $L^1(S^3)$ и имеет место неравенство $\sup_N \left\| \sum_{n=1}^{N} Y_n[g,(\theta,\varphi)] \right\|_{L^1(S^3)} << 3\|g\|_{L^1(S^3)} \le 12 \| f \|$.