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Mathematics

TWO SIDE ESTIMATES FOR THE DOUBLE OBSTACLE PROBLEM

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Two obstacle problem in abstract form is considered in this paper. We prove two side estimates for the solution of this problem.

Keywords: double obstacle problem, two side estimates, Lewy–Stampacchia inequalities, T-monotonicity.

Introduction. In this paper we consider the following problem.

Let X be a real reflexive Banach space endowed with partial order $\leq$, which is a vector lattice under this order, $V \subset X$ is a closed subspace and sublattice, let A be a mapping from X to V' – the dual space of V, $L \in V'$, $\varphi, \psi \in X$. Our aim is to find $u \in K_\psi^\varphi$ such that

$$\langle Au - L, v - u \rangle \geq 0 \quad \forall v \in K_\psi^\varphi,$$

where

$$K_\psi^\varphi = \{u \in V : \ \psi \leq v \leq \varphi\}$$

and $\langle \cdot,\cdot \rangle$ is usual pairing between V and V'.

The main result of this paper is the following statement (see Theorem 3).

If the operator A is *coercive*, *strictly T-monotone* and *hemi-continuous* acting from X to V' (for the definitions we refer the reader to section 4), then the following two side estimates hold:

$$L \wedge A\varphi \leq Au \leq L \vee A\psi \quad \text{in} \quad V^*,$$

where u is the unique solution of the above problem. Here V^* is the *order dual* of V (for the definitions see the sections 3 and 4).

For this kind of estimates for one obstacle problem see [1, 2].

Preliminary background. Let E be a linear topological space and C be a nonempty closed and convex subset of E.

Definition 1. A function $g : C \times C \to R$ is called *monotone*, if $g(v,v) \leq 0$ for every $v \in C$ and

$$g(v,w) + g(w,v) \geq 0 \ \ \forall v, w \in C. \tag{1}$$

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g is called *strictly monotone*, if it is monotone and strict inequality holds in (1), if $v \neq w$.

Note that for a monotone function g we have $g(v,v)=0$ for every $v \in C$.

Definition 2. A function $f: E \to R$ is called *lower (upper) semi-continuous* at $t_0 \in E$, if

$$f(t_0) \leq \liminf_{t \to t_0} f(t) \quad \left(f(t_0) \geq \limsup_{t \to t_0} f(t) \right).$$

Definition 3. A function $g: C \times C \to R$ is called *hemi-continuous*, if

i. $g(v,v) \leq 0$ for every $v \in C$;

ii. for every $v, w \in C$ the function $g_{v,w}:[0,1] \to R$ defined by $g_{v,w}(t) = g(v + t(w-v), w)$ is lower semi-continuous at $t = 0$.

Remark 1. We recall that a map $A: C \to E'$, where C is a convex subset of E, is said to be *hemi-continuous,* if it is continuous from the line segment of C to the weak topology of E'. Clearly, if A is hemi-continuous from C to E', then the function g, given by $g(v,w) = \langle Av, v-w \rangle$ $\forall v, w \in C$, is hemi-continuous.

The next theorem is a classical result about the existence and uniqueness of one obstacle problem for monotone and hemi-continuous mappings. For the proof of this theorem we refer to [2].

Theorem 1. Let E be a Hausdorff topological space, C be a closed convex subset of E, $\psi: E \to R$ be convex and lower semi-continuous and let $g: C \times C \to R$ be a monotone, hemi-continuous mapping such that $g(v,\cdot)$ is convex and upper semi-continuous for each $v \in C$. Let us assume also that there is a compact subset B of E and $w_0 \in B \cap C$, such that

$$\psi(v) + g(v, w_0) > \psi(w_0) \text{ for all } v \in C \setminus B.$$

Then the set of all solutions v of the problem

$$\begin{cases} v \in C, \\ \psi(v) + g(v,w) \leq \psi(w), \ \forall w \in C \end{cases} \tag{2}$$

is a non-empty convex compact subset of $B \cap C$. Moreover, if g is strictly monotone, then the problem (2) has unique solution.

Ordered Banach spaces. Let X be a real Banach space, and $\leq$ be a partial order relation in X induced by a closed positive cone

$$P = \{ v \in X : v \geq 0 \}. \tag{3}$$

We assume that X is a vector *lattice* with this ordering. That is, any two vectors u and v of X have a common least upper bound, denoted by $u \vee v$, and a common greatest lower bound denoted by $u \wedge v$. Then every vector $v \in X$ can be decomposed as

$$v = v^+ - v^-, \tag{4}$$

where $v^+ = v \vee 0$ and $v^- = -v \wedge 0$ are the *positive* and *negative* parts of v, respectively. In other words, the positive cone (3) is *generating*, i.e. $P - P = X$.

The decomposition (4) and its generalization

$$v+w=v\vee w+v\wedge w,\quad v,w\in X,\tag{5}$$

are obtained from the identity

$$v\vee w+z=(v+z)\vee(w+z),\quad v,w,z\in X.\tag{6}$$

In particular, (5) follows from (6) if we take $z=-v-w$ (note that $(-w)\vee(-v)=-(w\wedge v)=-(v\wedge w)$). From (6) we also obtain the identities

$$v\vee w=v+(w-v)^{+}=w+(v-w)^{+},\tag{7}$$

$$v\wedge w=v-(v-w)^{+}=w-(w-v)^{+},\tag{8}$$

that will be often used below.

Recall that a lattice X is said to be complete, if every subset $Z\subset X$ with an upper bound possesses a least upper bound, denoted by $\vee Z$. If V is a subspace of X, we say that V is a sublattice of X, if for any two vectors v and w of V, the elements $v\wedge w$ and $v\vee w$, formed in X also belong to V.

For such V we denote the order dual of V by V^{*}, which is the (closed) subspace of the dual space V', generated by positive cone

$$P'=\left\{v'\in V':\langle v',v\rangle\geq 0\ \forall v\in P\right\},\tag{9}$$

the pairing appearing in (9) being the duality pairing of V and V'. In other words,

$$V^{*}=P'-P'.$$

The order dual V^{*} will not, in general, coincide with the whole dual space V' (see the example below).

V^{*} is a vector lattice under the *dual ordering*, that is, the partial ordering induced by the (closed) positive cone (9). In particular, for arbitrary v' and w' in V^{*} we can define $v'\wedge w'$ and $v'\vee w'$, both elements of V^{*}, and we have as above the identities

$$v'\vee w'=v'+(w'-v')^{+}=w'+(v'-w')^{+},$$

$$v'\wedge w'=v'-(v'-w')^{+}=w'-(w'-v')^{+}.$$

Example. In applications to variational and quasi-variational inequalities for linear second order (elliptic) PDE in divergence form, the space X usually will be the Sobolev space $H^{1}(\Omega)$, where Ω is a smooth bounded open subset of R^{n}, and V will be either the Sobolev space $H_{0}^{1}(\Omega)$, or any closed subspace of $H^{1}(\Omega)$ such that

$$H_{0}^{1}(\Omega)\subset V\subset H^{1}(\Omega).\tag{10}$$

Under the following order,

$$u\leq v\Leftrightarrow u(x)\leq v(x)\ \text{a.e.}\ x\in\Omega,\tag{11}$$

the space $X=H^{1}(\Omega)$ is a vector lattice, and the positive cone (3) is closed (see [3]). Note that $H^{1}(\Omega)$ can be identified with a sublattice of $L^{2}(\Omega)$, which is complete lattice with respect to (11) (see [4]). The subspace V satisfying (10) is usually defined in terms of the boundary conditions for the problem at hand. Moreover, one should not forget that V was a sublattice of $X=H^{1}(\Omega)$. Particularly, the subspace $H_{0}^{1}(\Omega)$ satisfies the Dirichlet's boundary conditions.

If $V = H_0^1(\Omega)$, then $V' = H^{-1}(\Omega)$. In this case $v' \in V'$ is a positive element for the dual ordering, if and only if v' is a positive distribution. Hence v' is a positive (Radon) measure in Ω, belonging to $H^{-1}(\Omega)$. The order dual of $H_0^1(\Omega)$ is thus the closed subspace of all distributions in $H^{-1}(\Omega)$, which may be represented as the difference of two positive measures in Ω.

T-monotone operators. Let X be a real reflexive Banach space, which is a vector lattice for the partial ordering induced by a closed positive cone (3), and let X' be the dual space of X.

Definition 4. We say that the operator $A : X \to X'$ is

a) *monotone*, if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \text{for all } u, v \in X, \tag{12}$$

b) *strictly monotone,* if it is monotone and strict inequality holds in (12) when $u \neq v$,

c) *hemi-continuous*, if the map $t \mapsto \langle A(u + tv), w \rangle$ is continuous on $[0,1]$ for all $u, v, w \in X$,

d) *coercive*, if there exists a $w_0 \in X$ such that

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u - w_0 \rangle}{\|u\|} = +\infty, \quad u \in X.$$

Assume that we are given a closed subspace V of X, which is a sublattice in X. We consider the map $A : X \to V'$ from the space X to the dual V' of V.

Definition 5. We will say that A is *T-monotone*, if

$$\langle Au - Av, (u - v)^+ \rangle \geq 0 \tag{13}$$

for every $u, v \in X$ such that $(u - v)^+ \in V$. A is said to be *strictly T-monotone*, if it is T-monotone and if $(u - v)^+ = 0$ whenever the equality holds in (13).

The pairing in (13), as all pairings below, is the duality pairing between V and its dual space V'.

Lemma 1. If the operator $A : X \to V'$ is T-monotone (strictly T-monotone), then its restriction to V is a monotone (strictly monotone) operator from V to V'.

For the proof of this Lemma we refer to [2].

Example. Let Ω be a bounded subset in R^n and

$$a(u, v) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j} v + c(x) uv \Big) dx \tag{14}$$

with $a_{ij}, b_j, c \in L^\infty(\Omega)$, $c(x) \geq 0$ a.e. in Ω, and suppose there is a constant $\gamma_0 > 0$ such as

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma_0 |\xi|^2 \quad \text{a.e. } x \in \Omega, \forall \xi \in R^n.$$

Let V be any closed subspace of the Sobolev space $H^1(\Omega)$, satisfying (10), such that for some $\gamma > 0$

$$a(v, v) \geq \gamma \|v\|^2_{H^1(\Omega)} \quad \forall v \in V. \tag{15}$$

If $V = H_0^1(\Omega)$, (15) follows from the above assumptions on the coefficients of the form. If $V = H^1(\Omega)$, then (15) still holds provided

$$c(x) \geq c_0 > 0 \ \text{ a.e. } x \in \Omega.$$

The identity

$$\langle Au, v \rangle = a(u,v), \quad u \in H^1(\Omega), v \in V$$

defines a (linear) strictly T-monotone operator from $X = H^1(\Omega)$ to V'.

It is a consequence of the following property of the form (14):

$$a(w^+, w^-) = 0 \ \text{ for every } \ w \in H^1(\Omega). \tag{16}$$

In fact we have

$$\langle Au - Av, (u-v)^+ \rangle = a(u-v, (u-v)^+)$$

therefore, due to (16) and (15),

$$a(u-v,(u-v)^+) = a((u-v)^+,(u-v)^+) \geq \gamma \left\| (u-v)^+ \right\|_V^2$$

whenever $(u-v)^+ \in V$. The latter assumption is provided, for example, in case $V = H_0^1(\Omega)$, if $u, v \in H^1(\Omega)$, $u - v \leq 0$ a.e. on the boundary Γ of Ω.

Note that under the above assumptions, the restriction of A to V is, by (15), a coercive continuous linear operator from V to V'.

For more examples of T-monotone operators we refer to [5].

Two side estimates.

Theorem 2. Let X be a real reflexive Banach space, C be a convex, closed subset of X, A be a coercive, monotone and hemi-continuous mapping from C to the dual X' of X. Then for every functional $L \in X'$ the set of all v satisfying

$$\begin{cases} v \in C \\ \langle Av, v-w \rangle \leq \langle L, v-w \rangle \quad \forall w \in C \end{cases} \tag{17}$$

is a non-empty bounded, closed, convex subset of C. Moreover, if A is strictly monotone then the above problem has unique solution.

Proof. Apply Theorem 1 with $E = X$ with the weak topology, $\psi = -L$ (X is reflexive) and $g(v,w) = \langle Av, v-w \rangle$, $v, w \in C$. Since A is monotone and hemi-continuous, so g is also monotone and hemi-continuous (see Remark 1). Since A is coercive too, then g satisfies to the following condition: there is a compact subset B of $E = X$ and a $w_0 \in B \cap C$, such that

$$\psi(v) + g(v, w_0) > \psi(w_0) \ \text{ for all } \ v \in C \setminus B.$$

We can obtain this by taking w_0 as the vector appearing in definition 4.d) and $B = \{ v \in X : \|v\| \leq R \}$ with $R > 0$ sufficiently large.

If A is strictly monotone, so is g, hence the problem (17) has unique solution.

Remark 2. For a real ordered Banach space V and for elements $\varphi, \psi \in V$, we define the set

$$K_\psi^\varphi = \{u \in V : \psi \le u \le \varphi\}.$$

It is easy to see that K_ψ^φ is a closed convex subset of V. We consider the following problem: find $u \in K_\psi^\varphi$ such that

$$\langle Au - L, v - u\rangle \ge 0, \quad \forall v \in K_\psi^\varphi.$$

If A and L are as in the previous Theorem then this problem has unique solution.

Remark 3. For $L_1, L_2 \in V'$ we say $L_1 \ge L_2$, if $L_1 - L_2 \in P'$.

Theorem 3 (Two side estimates). Let X be a real reflexive ordered Banach space, V – closed subspace of X, which is a sublattice of X, A be a coercive, strictly T-monotone and hemi-continuous mapping from X to V'. Let two elements $\varphi, \psi \in X$, $\psi \le \varphi$ be given, and let $L \in V'$. Suppose that

$$\exists \Lambda \in V' \text{ such that } \Lambda \ge L \text{ and } \Lambda \ge A\psi \text{ in } V', \tag{18}$$

$$\exists \lambda \in V' \text{ such that } \lambda \le L \text{ and } \lambda \le A\varphi \text{ in } V', \tag{19}$$

and

$$(\psi - v)^+ \in V \quad \forall v \in V, \tag{20}$$

$$(v - \varphi)^+ \in V \quad \forall v \in V. \tag{21}$$

Then if u is the solution of

$$u \in K_\psi^\varphi : \quad \langle Au - L, v - u\rangle \ge 0 \quad \forall v \in K_\psi^\varphi, \tag{22}$$

where

$$K_\psi^\varphi = \{u \in V : \psi \le u \le \varphi\}, \tag{23}$$

then one has the following two side estimate

$$\lambda \le Au \le \Lambda \text{ in } V'. \tag{24}$$

In particular, if $L, A\varphi, A\psi$ belong to the order dual V^* of V, one has also $Au \in V^*$ and (24) becomes

$$L \wedge A\varphi \le Au \le L \vee A\psi \text{ in } V^*. \tag{25}$$

Proof. This last assertion holds, since, if $L, A\varphi, A\psi \in V^*$, one can take $\lambda = L \wedge A\varphi = L - (L - A\varphi)^+$ (see (8)) and $\Lambda = L \vee A\psi = L + (A\psi - L)^+$ (see (7)) in (24).

To prove the upper bound of (24) we recall the Theorem 2 and consider the unique solution $z \in V$ of auxiliary variational inequality

$$z \le u : \quad \langle Az - \Lambda, w - z\rangle \ge 0 \quad \forall w \in V, w \le u. \tag{26}$$

It is enough to prove that $z = u$, since then taking $w = u - v$ in (26) for an arbitrary $v \ge 0$, we obtain

$$Au - \Lambda = Az - \Lambda \le 0 \text{ in } V'.$$

To prove that $z = u$, let us first prove that $z \ge \psi$. Due to (20) and taking $w = z + (\psi - z)^+ = \psi \vee z \le u$ in (26), we get

$$\langle \Lambda - Az, (\psi - z)^+\rangle \le 0.$$

Hence, since $\Lambda \geq A\psi$, one gets

$$\left\langle A\psi - \Lambda, (\psi - z)^+ \right\rangle + \left\langle \Lambda - Az, (\psi - z)^+ \right\rangle \leq 0 ,$$

which by the strict T-monotonicity of A implies that $(\psi - z)^+ = 0$. This means that $z \geq \psi$.

Let us now prove that $z \geq u$. Since u solves (22) and in (26) $z \leq u$, then $z \leq \varphi$. In other words $z \in K_\psi^\varphi$. Take $w = z \vee u = z + (u - z)^+ \leq u$ in (26) and $v = u \wedge z = u - (u - z)^+ \geq \psi$ in (22). Summing the obtained inequalities and using $\Lambda \geq L$ from (18), we have

$$\left\langle Au - Az, (u - z)^+ \right\rangle \leq \left\langle L - \Lambda, (u - z)^+ \right\rangle \leq 0,$$

and, since A is strictly T-monotone, it follows that $(u - z)^+ = 0$ and so $z \geq u$.

So, the unique solution u of (22) is also the unique solution of (26). We already know, that this implies the upper bound in (24).

For obtaining the lower bound in (24), we recall again the Theorem 2 and consider the unique solution $z \in V$ of auxiliary variational inequality

$$z \geq u: \ \left\langle Az - \lambda, w - z \right\rangle \geq 0 \ \ \forall w \in V, \ w \geq u. \qquad (27)$$

The steps are similar as above. It is enough to prove that $z = u$, since taking $w = u + v$ in (27) for an arbitrary $v \geq 0$, it will follow that

$$\lambda - Au = \lambda - Az \leq 0 \ \text{ in } \ V'.$$

To prove that $z = u$, let us first prove that $z \leq \varphi$. Due to (21) and taking $w = z - (z - \varphi)^+ = z \wedge \varphi \geq u$ in (27), we get

$$\left\langle Az - \lambda, (z - \varphi)^+ \right\rangle \leq 0.$$

Hence, since $\lambda \leq A\varphi$, one gets

$$\left\langle \lambda - A\varphi, (z - \varphi)^+ \right\rangle + \left\langle Az - \lambda, (z - \varphi)^+ \right\rangle \leq 0,$$

which, due to the strict T-monotonicity of A , implies that $(z - \varphi)^+$ =0. This means that $z \leq \varphi$.

Let us now prove that $z \leq u$. Since u is the solution for (22) and in (27) $z \geq u$, one gets that $z \geq \psi$. In other words, $z \in K_\psi^\varphi$. Take $w = z \wedge u = z - (z - u)^+ \geq u$ in (27) and $v = u \vee z = u + (z - u)^+ \leq \varphi$ in (22). Substracting the obtained inequalities and using $\lambda \leq L$ from (19), we have

$$\left\langle Az - Au, (z - u)^+ \right\rangle \leq \left\langle \lambda - L, (z - u)^+ \right\rangle \leq 0,$$

and, since A is strictly T-monotone, it follows that $(z - u)^+ = 0$ and so $z \leq u$.

So, we have proved that the unique solution u of (22) is also the unique solution of (26) (we have already known that this implies the upper bound in (24)) and the unique solution of (27), which, as we see, implies the lower bound in (24).

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REFERENCES

1. **Rodrigues J.F.** Obstacle Problems in Mathematical Physics. In: North-Holland Mathematics Studies, Notas de Matematica. 1987, v. 114.
2. **Mosco U.** Implicit Variational Problems and Quasi Variational Inequalities, Lecture notes in Mathematics. Bruxells: Springer Verlag, 1976, № 543, p. 83–156.
3. **Littman W., Stampacchia G., Weinberger H.** Ann. Scuola Norm. Sup. Pisa, 1963, v. 17, p. 43–77.
4. **Schaeffer H.** Topological vector spaces. New York: Mac-Millan ed., 1966.
5. **Lions J.L.** Quelques methodes de resolution des problemes aux limites non lineares. Paris: Dunod and Gauthiers-Cillard ed., 1969.

Ռ. Ռ. Թեյմուրազյան

Երկու խոչընդոտով խնդրի լուծման երկկողմանի գնահատականներ

Աշխատանքում քննարկվում է երկու խոչընդոտով խնդիրը՝ ընդհանուր դրվածքով: Որոշակի պայմանների առկայության դեպքում ապացուցվում են երկկողմանի գնահատականներ այդ խնդրի լուծման համար:

Р.Р. Теймуразян.

Двусторонние оценки для задачи с двумя препятствиями

В работе рассматривается задача с двумя препятствиями в абстрактной постановке. Доказываются двусторонние оценки для решения этой задачи при некоторых предположениях.