PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2009, № 1, p. 24–28

Mathematics

ON INCREASING HAZARD RATE OF WAITING TIME IN THE $M\,|\,G\,|\,1\,|\,\infty$ MODEL

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In the present paper for the sum of random number of independent identically distributed random variables the hazard rate increase is proven under the summanders hazard rate increase condition. With the help of obtained result the stationary waiting time hazard rate increase in the $M\,|\,G\,|\,1\,|\,\infty$ model is established under the service time hazard rate increase contition.

***Keywords*:** $M\,|\,G\,|\,1\,|\,\infty$ model, waiting time, increasing hazard rate.

1°. Introduction. Let B be a distribution function (DF) of non-negative random variable (RV). Under the condition $B(+0)=0$ the DF B is reffered as DF with *increasing hazard rate* (IHR DF), if for any $x \in R^+ = (0,+\infty)$ the function

$$\left(B(x+t)-B(t)\right)\big/\left(1-B(t)\right) \tag{1}$$

is non-decreasing with respect to $t \in R^+$.

We conserve the introduction definition also for DF F having a jump at zero: $F(x)=\alpha+(1-\alpha)B(x)$, $x\in R^+$, $\alpha\in[0,1)$, $B(+0)=0$. Then simultaneously DFs F and B are or not IHR DFs, because (see (1))

$$\left(F(x+t)-F(t)\right)\big/\left(1-F(t)\right)=\left(B(x+t)-B(t)\right)\big/\left(1-B(t)\right),\ \ x\in R^+,\ \ t\in R^+. \tag{2}$$

Let ξ_1 and ξ_2 be independent RVs with IHR DFs B_1 and B_2. It is known that under the condition $B_1(+0)=B_2(+0)=0$ the DF of RV $\zeta=\xi_1+\xi_2$, i. e. $B_1 * B_2$, where $*$ presents the sign of convolution, is an IHR DF (see [1]).

Let $\{\xi_n\}$ be a sequence of non-negative independent identically distributed (IID) RVs with the IHR DF B, and ν be an entite valued random index, which doesn't depend on $\{\xi_n\}$:

$$P(\nu=n)=\alpha_n,\ n=0,\ 1,\ 2,\dots,\ \sum_{n\ge 0}\alpha_n=1.$$

Here P is a sign of probability. Denote by

$$\zeta \overset{d}{=} \xi_1+\xi_2+\cdots+\xi_\nu \tag{3}$$

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the sum of random number IID RVs, and by F the DF of RV ζ.

The symbol d in (3) means that DFs of both sides of stochastic equality are coincided.

In other words, by the formula of total probability,

$$F(x)=\sum_{n\geq 0}\alpha_n\, B^{n*}(x)\,,\ x\in R^+\,, \tag{4}$$

where B^{n*} is n fold convolution of B by itself and $B^{0*}(x)\equiv 1$.

Theorem 1. If B IHR DF given by formula (4), then (see also (3)) F is the IHR DF.

Let us consider the $M\,|\,G\,|\,1\,|\,\infty$ quene model with FIFO service discipline (see, for instance, [2]). Namely in a single server quene with infinite waiting room the Poisson stream of customers with intensity $a\in R^+$ arrive. The service times of customers are independent, do not depend at the arrival process and have DF B, $B(+0)=0$.

The number $\rho_1=a\beta_1$ is called *traffic intensity* of the model, where

$$\beta_1=\int_0^\infty\big(1-B(u)\big)du\,. \tag{5}$$

Denote by W the stationary waiting time DF. The necessary and sufficient condition for W 's existence is $\rho_1<1$ (see [2]).

With the help of Theorem 1 the following statement is established.

Theorem 2. If B is an IHR DF, then in the $M\,|\,G\,|\,1\,|\,\infty$ model with FIFO discipline W is an IHR DF.

The proof of Theorem 2 uses the following auxilary statement. Let

$$\hat{B}(x)=\frac{1}{\beta_1}\int_0^x\big(1-B(u)\big)du \quad \text{(see (5))}. \tag{6}$$

It is easy to see that $\hat{B}$ is a DF.

Lemma. If B is the IHR DF and $\beta_1\in R^+$, where β_1 is defined by equality (5), then $\hat{B}$ is an IHR DF.

Lemma is proven in Section 2°, Theorems 1 and 2 – in Appendix.

2°. Auxilary Statement. First of all let us prove Lemma under the condition: the IHR DF B in the model $M\,|\,G\,|\,1\,|\,\infty$ has a density, say $b(x)$. Then the condition (1) is equivalent to the following one (see [1]):

The function $\big(b(t)/\big(1-B(t)\big)\big)$ *non-decreases as* t *increases.* (7)

Due to (6) the DF $\hat{B}$ has the following density

$$\hat{b}(t)=\frac{1}{\beta_1}\big(1-B(t)\big)\,,\ t\in R^+\,.$$

That is why the condition (7) for $\hat{B}$ may be written in the form: the function

$$\frac{\hat{b}(t)}{1-B(t)}=\frac{1-B(t)}{\beta_1\left(1-\beta_1^{-1}\int_0^t\big(1-B(u)\big)du\right)}=\frac{\int_t^\infty b(u)du}{\int_t^\infty\big(1-B(u)\big)du}=$$

$$=\left(\int_t^{\infty}\left(1-B(u)\right)g(u)du\Big/\int_t^{\infty}\left(1-B(u)\right)du\right) \tag{8}$$

doesn't decrease when t increases. Here we denote

$$g(t)=\left(b(t)/\left(1-B(t)\right)\right),\ t\in R^+. \tag{9}$$

In order to verify the validity of condition (8) one needs to show that for $0<t_1<t_2<+\infty$ the inequality holds

$$\left\{\left(\int_{t_1}^{t_2}+\int_{t_2}^{+\infty}\right)\left(1-B(u)\right)g(u)du\right\}\Big/\left\{\int_{t_2}^{+\infty}\left(1-B(u)\right)du\right\}\le$$

$$\le\left\{\left(\int_{t_1}^{t_2}+\int_{t_2}^{+\infty}\right)\left(1-B(u)\right)du\right\}\Big/\left\{\int_{t_2}^{+\infty}\left(1-B(u)\right)g(u)du\right\}$$

or

$$\frac{\int_{t_1}^{t_2}\left(1-B(u)\right)g(u)du}{\int_{t_1}^{t_2}\left(1-B(u)\right)du}\le\frac{\int_{t_2}^{+\infty}\left(1-B(u)\right)g(u)du}{\int_{t_2}^{+\infty}\left(1-B(u)\right)du}. \tag{10}$$

Due to (7), the function $g(t)$ given by formula (9) doesn't decrease with respect to t. Therefore for $0<t_1<t_2<+\infty$ the following inequalities hold

$$\int_{t_1}^{t_2}\left(1-B(u)\right)g(u)du\le g(t_2)\int_{t_1}^{t_2}\left(1-B(u)\right)du,$$

$$\int_{t_2}^{+\infty}\left(1-B(u)\right)g(u)du\ge g(t_2)\int_{t_2}^{+\infty}\left(1-B(u)\right)du.$$

From here and (10) the statement follows in the case, when DF B has a density.

Now let us prove Lemma in general case. Let $\lambda\in R^+$. The DF

$$B_\lambda(x)=B(x)*\left(1-e^{-\lambda x}\right),\ x\in R^+, \tag{11}$$

as a convolution of IHR DFs with $B(+0)=0$, $\left(1-e^{-\lambda x}\right)\Big|_{x=0}=0$, is an IHR DF, $B_\lambda(+0)=0$ and $B_\lambda(x)$ has a density. Hence due to above proved the DF (see (11))

$$\hat{B}_\lambda(t)=\frac{1}{\beta_{\lambda 1}}\int_0^t\left(1-B(u)*\left(1-e^{-\lambda u}\right)\right)du, \tag{12}$$

where

$$\beta_{\lambda 1}=\beta_1+\frac{1}{\lambda} \tag{13}$$

is an IFR DF. Let us show that for all $t\in R^+$

$$\lim_{\lambda\to+\infty}\hat{B}_\lambda(t)=\hat{B}(t), \tag{14}$$

where $\hat{B}$ is defined by formula (6).

The limity relationship (14) follows from the continute Theorem for the Laplas–Steltjes Transform. Indeed, for $s \geq 0$

$$\hat{\beta}_\lambda(s) = \int_0^\infty e^{-sx} d\hat{B}_\lambda(x) = \frac{1}{\beta_{\lambda 1}} \int_0^\infty e^{-sx}\left(1 - B(x) * \left(1 - e^{-\lambda x}\right)\right)dx = \frac{1 - \beta(s)\dfrac{\lambda}{s+\lambda}}{s\beta_{\lambda 1}} . \quad (15)$$

For $\lambda \to +\infty$ we get that the right-hand-side of (15) tends to the function

$$\hat{\beta}(s) = \frac{1 - \beta(s)}{s\beta_1},$$

where (13) was used. Since $\hat{\beta}(s)$ is the Laplas–Steltjes Transform for DF $\hat{B}$, therefore (14) is proven.

Taking into account that the limit of IHR DFs is an IHR DF, if the limit is DF, which is a consequence of (1) due to (14) we conclude that $\hat{B}$ is an IHR DF.

3°.Appendix. First of all let us prove Theorem 1 in particular case

$$P(\nu = k) = \alpha_k ,\ k = 0,\ 1,\ 2, \ldots,\ n ,\ \sum_{k=0}^{n} \alpha_k = 1 , \quad (16)$$

by induction with respect to n. For $n = 1$ we have

$$\alpha_0 + \alpha_1 B(x) = \alpha_0 + \left(1 - \alpha_0\right) B(x) = F(x) .$$

If $\alpha_0 = 0$, then B is an IHR DF. Let $\alpha_0 > 0$. Then F is an IHR DF because of (2), which implies the basement of the induction.

Let the statement is true for any collection $(\alpha_1,\ \alpha_2, \ldots,\ \alpha_n)$ and for IHR DF B, where (16) takes place. Let us prove that the statement is true for index $n+1$. Consider

$$\sum_{k=0}^{n+1} \alpha_k B^{k*}(x) ,\ x \in R^+ ,$$

with an arbitrary collection $(\alpha_1,\ \alpha_2, \ldots,\ \alpha_n,\ \alpha_{n+1})$, where $\sum_{k=0}^{n+1} \alpha_k = 1$.

If $\alpha_0 = 0$, then

$$\sum_{k=0}^{n+1} \alpha_k B^{k*}(x) = B(x) * \sum_{k=1}^{n+1} \alpha_k B^{(k-1)*}(x) . \quad (17)$$

Since B and due to the induction assumption $\sum_{k=1}^{n+1} \alpha_k B^{(k-1)*}(x)$ is an IHR DF, therefore the convolution (17) is an IHR DF. Indeed in order to establish this fact let us give another definition of the IHR DF.

The function $p(x), x \in R^1 = (-\infty,\ +\infty)$, is called a *Poja density,* if $p(x) \geq 0$ and $\begin{vmatrix} p(x_1 - y_1) ,\ p(x_1 - y_2) \\ p(x_2 - y_1) ,\ p(x_2 - y_2) \end{vmatrix} \geq 0$ for any $-\infty < x_1 \leq x_2 < +\infty$, $-\infty < y_1 \leq y_2 < +\infty$.

It was proven in [1] that B is an IHR DF, if $1 - B$ is a Poja density under the condition $B(+0) = 0$. Moreover, the fact that $B_1 * B_2$ is an IHR DF, when B_1 and B_2 are IHR DFs under the condition $B_1(+0) = B_2(+0) = 0$ was proven in [1] with the help of the last criterion. Analyzing the proof, we convinced that the condition $B_1(+0) = B_2(+0) = 0$ may be omitted.

Now let $\alpha_0 > 0$. We have

$$\sum_{k=0}^{n+1} \alpha_k B^{k*}(x) = \alpha_0 + B(x) * \sum_{k=1}^{n+1} \alpha_k B^{(k-1)*}(x) =$$

$$= \alpha_0 + (1-\alpha_0) B(x) * \sum_{k=1}^{n+1} \frac{\alpha_k}{1-\alpha_0} B^{(k-1)*}(x), \ x \in R^+ . \quad (18)$$

Since $\sum_{k=1}^{n+1} \frac{\alpha_k}{1-\alpha_0} = 1$, therefore, due to induction assumption and due to the fact that B is an IHR DF, we conclude $\sum_{k=1}^{n+1} \frac{\alpha_k}{1-\alpha_0} B^{(k-1)*}(x)$ is an IHR DF.
From here due to (18) we come to the desired statement.

Now, consider the general case (4). Introduce the following sequence of DFs

$$F_n(x) = \sum_{k=0}^{n} \alpha_k B^{k*}(x) + \left(\sum_{k>n} \alpha_k \right) B^{(n+1)*}(x), \ x \in R^+, \ n = 0, 1, 2, \ldots$$

Due to the obove proved $\{F_n(x)\}$ is a sequence of IHR DFs, if B is an IHR DF.

Since $\lim_{n\to+\infty} F_n(x) = F(x)$, $x \in R^+$, therefore, F is an IHR DF.

Theorem 1 is proven.

Pass to the *proof* of Theorem 2.

By Cohen's formula for the $M|G|1|\infty$ model (see [3])

$$W(x) = (1-\rho_1) \sum_{n\geq 0} \rho_1^n \hat{B}^{n*}(x), \ x \in R^+, \quad (19)$$

where $\hat{B}$ is defined by formula (6). According to (19), the stationary waiting time w may be represented in the form (compare to (3))

$$w \overset{d}{=} \hat{\beta}_1 + \hat{\beta}_2 + \cdots + \hat{\beta}_\nu ,$$

where $\{\hat{\beta}_n\}$ is a sequence of IID RVs with DF $\hat{B}$, and ν is a geometric index, which doesn't depend on $\{\hat{\beta}_n\}$, i. e. $P(\nu = n) = (1-\rho_1)\rho_1^n$, $n = 0, 1, 2, \ldots$

Since B is an IHR DF, therefore, due to Lemma $\hat{B}$ is an IHR DF. Now applying Theorem 1 to (19), we come to Theorem 2.

Received 27.12.2008

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Հ. Ս. Հարությունյան, Հ. Հ. Իգիթյան

$\boldsymbol{M\,|\,G\,|\,1\,|\,\infty}$ մոդելում սպասման ժամանակի ծերանալու մասին

Աշխատանքում ապացուցված է պատահական թվով անկախ միատեսակ բաշխված պատահական մեծությունների գումարի ծերանալը գումարելիների ծերանալու պայմանի դեպքում: Ստացված արդյունքն օգտագործված է $\boldsymbol{M\,|\,G\,|\,1\,|\,\infty}$ մոդելի ստացիոնար սպասման ժամանակի ծերանալու ապացույցի ժամանակ, երբ սպասարկման ժամանակները ծերացող են:

Г. С. Арутюнян, Г. О. Игитян.

О старении времени ожидания в модели $\boldsymbol{M\,|\,G\,|\,1\,|\,\infty}$

В статье для суммы случайного числа независимых одинаково распределенных случайных величин доказано старение функции распределения при старении функции распределения слагаемых. На основе полученного результата установлено старение функции распределения стационарного времени ожидания в модели $M\,|\,G\,|\,1\,|\,\infty$ при условии старения функции распределения времени обслуживания.