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A REMARK ON ASYMPTOTIC PROPERTY OF COMMUTATORS

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In the present paper the asymptotic variance of the classical Von-Neiman–Fuglede Theorem for elements of the complex Banach algebra is extended.

***Keywords*:** Banach algebra, hermitian, normal element, commutator.

It has been proved in [1], that if a, b, x are elements of the complex Banach algebra with unit element, whereas $[a, b] = ab - ba = 0$ and $\|\exp(ita)\| = o\left(|t|^{1/2}\right)$, $\|\exp(itb)\| = o\left(|t|^{1/2}\right)$, for real $t \to \pm\infty$ and $[a+ib, x]=0$, then $[a-ib, x]=0$.

Later in [2] it was shown that in the above mentioned result the condition $o\left(|t|^{1/2}\right)$ can not be replaced by $O\left(|t|^{1/2}\right)$, however here the weakening of the condition $[a,b]=0$ plays the central role.

In [3] the class $Gr(A)$ with weakening condition $[a,b]=0$ was introduced.

In the present paper we consider the asymptotic cases of these results.

Let A be a Banach algebra with unit element **1** over the field of complex numbers $\mathbb{C}$ (we assume that $\|\mathbf{1}\| = 1$ and $\|xy\| \le \|x\| \cdot \|y\|$ for all $x, y \in A$). $\mathbb{C}$ – linear functional, then $\varphi : A \to \mathbb{C}$ is called a "state", if $\|\varphi\| = \varphi(\mathbf{1}) = 1$.

The set $St(A)$ of all states forms $\sigma\left(A^*, A\right)$-compact, convex subset of the dual space A^*. Note that (see [3, 4]) the element $\boldsymbol{h} \in A$ is called "hermitian", if $\varphi(\boldsymbol{h}) \subset \mathbb{R}$ for all $\varphi(\boldsymbol{h}) \in St(A)$, which is equivalent to the condition $\|\exp(it\boldsymbol{h})\| = 1$ for all real t. The set of all hermitian elements $H(A)$ of the algebra A is a closed $\mathbb{R}$-linear subspace of the algebra A. Note that an element $a \in A$ is called hermitian-decomposable, if it allows a representation of the form $a = \boldsymbol{h} + i\boldsymbol{k}$, where $\boldsymbol{h}, \boldsymbol{k} \in H(A)$. Such a representation if exists is unique. The class of all hermitian-decomposable elements of algebra A is denoted by $H_{\mathbb{C}}(A)$, and it is

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closed, $\mathbb{C}$-linear subspace of A, which appears to be at the same time a Lie algebra with respect to commutator.

Let's choose a local convex topology τ on algebra A, satisfying the following properties: the mapping $(A,\|\cdot\|)\to(A,\tau)$ is continuous, the multiplication is separately τ-continuous. Note that standard topologies in algebras of operators do have these properties. Remember (see [3]), that an element $a\in A$ belongs to class $Gr(A)$, if there exists an element $b\in A$ such that

$$\max\left\{\left\|\exp(-\lambda b)\cdot\exp(\overline{\lambda}a)\right\|;\left\|\exp(-\overline{\lambda a})\cdot\exp(\lambda b)\right\|\right\}=o(|\lambda|^{1/2})\ \text{ for } |\lambda|\to\infty,\ \lambda\in\mathbb{C}.$$

Theorem 1. Let A be a complex Banach algebra with unit element, on which the above mentioned local-convex topology τ is defined. Then for each neighborhood $U\subset A$ of zero in topology τ, there exists a neighborhood $V\subset A$ of zero in the same topology τ, such that if $x\in A,\ \|x\|\le 1,\ a\in Gr(A),\ [a,x]\in V$, then $[b,x]\in U$.

Proof. Let q be a continuous algebraic semi-norm on $\{A,\tau\}$ and $\varepsilon>0$. We have to point out in the topology τ a neighborhood V of zero, such that if $\|x\|\le 1$ and $[a,x]=V$, then $[b,x]\in U$. We assume $q(x)\le\|x\|$ and $\|a\|\le 1,\ \|b\|\le 1$ for all $x\in A$.

Let φ be an arbitrary linear functional on A, and $|\varphi(x)|\le q(x)$ for all $x\in A$. Let consider the following entire function $f_\varphi(\lambda)=\varphi\left(\exp(-\lambda b)\cdot x\cdot\exp(\lambda b)\right)$, which can be represented as

$$f_\varphi(\lambda)=\varphi\Big(\exp(-\lambda b)\cdot\exp(\overline{\lambda}a)\cdot x\cdot\exp(-\overline{\lambda}a)\cdot\exp(\lambda b)\Big)-$$
$$-\varphi\Big(\exp(-\lambda b)\cdot\Big(\exp(\overline{\lambda}a)\cdot x-x\cdot\exp(\overline{\lambda}a)\Big)\cdot\exp(-\overline{\lambda}a)\exp(\lambda b)\Big).$$

Then

$$|f_\varphi(\lambda)|\le\left|\varphi\Big(\exp(-\lambda b)\cdot\exp(\overline{\lambda}a)\cdot x\cdot\exp(-\overline{\lambda}a)\cdot\exp(\lambda b)\Big)\right|+$$
$$+\left|\varphi\Big(\exp(-\lambda b)\cdot\Big(\exp(\overline{\lambda}a)\cdot x-x\cdot\exp(\overline{\lambda}a)\Big)\cdot\exp(-\overline{\lambda}a)\exp(\lambda b)\Big)\right|\le$$
$$\le q\Big(\exp(-\lambda b)\cdot\exp(\overline{\lambda}a)\cdot x\cdot\exp(-\overline{\lambda}a)\cdot\exp(\lambda b)\Big)+$$
$$+q\Big(\exp(-\lambda b)\cdot\Big(\exp(\overline{\lambda}a)\cdot x-x\cdot\exp(\overline{\lambda}a)\Big)\cdot\exp(-\overline{\lambda}a)\exp(\lambda b)\Big)\le$$
$$\le o(|\lambda|)+q([a,x])|\lambda|o(|\lambda|^{1/2})e^{2|\lambda|}.$$

Due to the Cauchy integral formula

$$f_\varphi'(0)=\frac{1}{2\pi i}\int_{\gamma_r}\frac{f_\varphi(\lambda)}{\lambda^2}d\lambda,$$

where γ_r is a circumference with radius r and with centre in the origin of coordinates. Since $f_\varphi'(0)=-\varphi([b,x])$, we have $|\varphi([b,x])|\le\frac{o(r)}{r}+$

$+q([a,x])o(\sqrt{r})e^{2r}$. Since $q([b,x])=\sup\{|\varphi([b,x])|:\varphi\in S\}$, where $S=\{\varphi\in A^*:|\varphi(x)|\le p(x)$ for all $x\in A\}$, we get

$$q([b,x])\le\frac{o(r)}{r}+q([a,x])o(\sqrt{r})e^{2r}.$$

Let's choose for $\varepsilon>0$ a radius r such that $\frac{o(r)}{r}<\frac{\varepsilon}{2}$ and $\delta<\frac{\varepsilon}{o(\sqrt{r})}e^{-2r}$. Therefore, if $q([a,x])<\delta$, then $q([b,x])<\varepsilon$, i.e. $[b,x]\in U$.

As a consequence we obtain the following results.

Theorem 2. Let A be a complex Banach algebra with unit element, and $a\in Gr(A)$. Then for every $\varepsilon>0$ there exists $\delta>0$, such that if $x\in A$ $\|x\|\le 1$ and $\|[a,x]\|<\delta$, then $\|[b,x]\|<\varepsilon$.

Proof. The proof follows from Theorem 1, if instead of topology τ one takes the topology of the norm on algebra A.

Corollary 1. Let A be a complex Banach algebra with unit element and $a\in Gr(A)\bigcap H_{\mathbb{C}}(A)$. Then for every $\varepsilon>0$ there exists $\delta>0$, such that if $x\in A,\|x\|\le 1$ and $\|[a,x]\|<\delta$, then $\left\|[a^+,x]\right\|<\varepsilon$.

Using Theorem 1, we can prove the following result.

Theorem 3. Let A be a complex Banach algebra with unit element, on which the above mentioned local-convex topology τ is defined. Then for each neighborhood $U\subset A$ of zero in the topology τ, there exists a neighborhood $V\subset A$ of zero in the same topology τ, such that if $x\in A$, $\|x\|\le 1$, $a\in Gr(A)$ and $ax-xb\in V$, then $bx-xa\in U$.

As in the proof of Theorem 1, for an arbitrary linear functional φ on A, for which $|\varphi(x)|<q(x)$ for all $x\in A$, we consider an entire function $F_\varphi(\lambda)=\varphi\left(\exp(-\lambda b)x\exp(\bar{\lambda}a)\right)$.

Then

$$F_\varphi(\lambda)=\varphi\left(\exp(-\lambda b)\exp(\lambda a)\cdot x\cdot\exp(-\bar{\lambda}b)\exp(\lambda a)\right)-$$
$$-\varphi\left(\exp(-\lambda b)\left(\exp(\bar{\lambda}a)\cdot x-x\cdot\exp(\bar{\lambda}b)\right)\exp(-\bar{\lambda}b)\exp(\lambda a)\right),$$

and we get a similar estimation

$$|F_\varphi(\lambda)|\le o(|\lambda|)+|\lambda|o(\sqrt{|\lambda|})q(ax-xb)e^{2|\lambda|}.$$

Finally we have

$$q(ax-xb)\le\frac{o(r)}{r}+q(ax-xb)o(\sqrt{r})e^{2r},$$

which proves the Theorem 3.

In the case, when the topology τ coincides with the topology of the norm on A, the following statement holds.

Theorem 4. Let A be a complex Banach algebra with unit element and $a \in Gr(A)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $x \in A$ $\|x\| \le 1$ and $\|ax - xb\| < \delta$, then $\|bx - xa\| < \varepsilon$.

Corollary 2. Let A be a complex Banach algebra with unit element and $a \in Gr(A) \bigcap H_{\mathbb{C}}(A)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $x \in A$ $\|x\| \le 1$ and $\|ax - xa^{+}\| < \delta$, then $\|a^{+}x - xa\| < \varepsilon$.

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Մեկ դիտողություն կոմուտատորների ասիմպտոտային հատկության վերաբերյալ

Տվյալ աշխատանքում ուժեղացվում է ֆոն Նեյմանի և Ֆուգլեդեի դասական թեորեմի ասիմպտոտային տարբերակը կոմպլեքս բանախյան հանրահաշվի տարրերի համար:

Одно замечание об асимптотическом свойстве коммутаторов

В настоящей работе усиливается асимптотический вариант классической теоремы фон Неймана–Фугледе для элементов комплексной банаховой алгебры.