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Mathematics

FINITE-ELEMENT METHOD FOR THE MODEL PSEUDOPARABOLIC EQUATION

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In the present paper the construction of the approximate solution to the initial-boundary value problem for the pseudoparabolic equation using finite-element method is considered. It is proved that the costructed sequence converges to the exact solution and error estimate is obtained.

Keywords: finite-element method, pseudoparabolic equations, monotone operators.

1. Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary Γ, $t > 0$. The following initial-boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = 0, \qquad x \in \Omega, & (1) \\ u\big|_{\Gamma} = 0, & (2) \\ u\big|_{t=0} = u_0(x), & (3) \end{cases}$$

where

$$L(u) = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(b_{ij}(x) \frac{\partial u}{\partial x_j} \right), \qquad M(u) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

$b_{ij}(x) = b_{ji}(x)$, $a_{ij}(x) = a_{ji}(x)$ ($i, j = 1, 2, \ldots, n$) are continuous functions in $\bar{\Omega}$ and inequality

$$\sum_{i,j=1}^{n} b_{ij}(x) \xi_i \xi_j \ge c_0 |\xi|^2$$

holds for $\forall x \in \bar{\Omega}$, $\forall \xi \in \mathbb{R}^n$, was investigated by R.A. Aleksandrian in [1]. The problem (1)–(3) for the case, when the operator L is linear, M is nonlinear and the operators L and M may degenerate, was investigated by G.S. Hakobyan, R.L. Shakhbaghyan (see [2]). The general case (when the operators L and M are nonlinear) are studied in [3], [4] and [5]. In [6] with the help of Galyorkin method the existence of solution for (1)–(3) was proved.

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In this paper we construct an approximate solution for the problem (1)–(3) using the finite-element method for the case $\Omega\in(0,1)\times(0,1)\subset\mathbb{R}^2$, $Lu=-\Delta u$, $Mu=-\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)$.

2. Definition. The function $u\in L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ is called a weak solution of the problem (1)–(3), if $u_t\in L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ and for $\forall v(x)\in \overset{\circ}{W}{}_2^1(\Omega)$ holds the equality

$$\int_\Omega \frac{\partial}{\partial t}(\nabla u)\cdot\nabla v\,dxdy+\int_\Omega\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial y}\right)dxdy=\int_\Omega f\,v\,dxdy. \qquad (*)$$

It was proved [1] that the equality (∗) has a unique solution. Now we construct an approximate solution to the problem (∗) using the finite-element method.

Suppose we partition the domain $\Omega=(0,1)^2$ into squares with side h with respect to x and y, and further divide the obtained squares into triangles (see Fig. 1)

$$x_{i+1}-x_i=h,\qquad i,j=1,2,\ldots,n,$$
$$y_{j+1}-y_j=h,\qquad h=\frac{1}{n}.$$

We construct piecewise linear functions $\varphi_{ij}(x,y)$ following the rule below

$$\varphi_{ij}(x_i,y_j=1),\ \varphi_{ij}(x_{i-1},y_j)=\varphi_{ij}(x_{i-1},y_{j+1})=\varphi_{ij}(x_i,y_{j+1})=\varphi_{ij}(x_{i+1},y_j)=$$
$$=\varphi_{ij}(x_{i+1},y_{j-1})=\varphi_{ij}(x_{i-1},y_{j-1})=0,$$

whereas they are linear inside the domain of any triangle. In the remaining triangles of the square $[0,1]\times[0,1]$ we assume $\varphi_{ij}(x,y)=0$. As result we get $N=(n-1)^2$ basis functions. Let us set $\omega_n=\{(ih,jh),\ i,j=1,2,\ldots,n-1\}$, and enumerate the points of the set ω_n (for example, $(ih,jh)=$ $=A_{\left(j-1,\sqrt{N}+i\right)}$), then the basis functions φ_{ij} will be renumerated as $\Psi_1,\Psi_2,\ldots,\Psi_N$, correspondingly. Thus by construction $\psi_k(A_r)=\delta_{kr}$ (k,r=1,2,…, n–1).

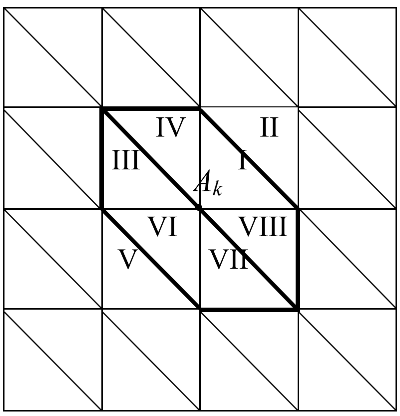

Fig. 1.

Denote by S_n the linear space generated by the functions ψ_i (i=1,2,…,N). Note that $\dim S_n=N$ and $S_n=\{v\in C(\bar\Omega),\ v$ is linear in every triangle and $v=0$ on $\partial\Omega\}$.

It is easy to see that $S_n\subset\overset{\circ}{W}{}_2^1(\Omega)$ is a subspace. To calculate $\frac{\partial\psi_k}{\partial x}$ and $\frac{\partial\psi_k}{\partial x}$ we use the following Table:

	I	II	III	IV	V	VI	VII	VIII
$\frac{\partial\psi_k}{\partial x}$	$-\frac{1}{h}$	0	$\frac{1}{h}$	0	0	$\frac{1}{h}$	0	$-\frac{1}{h}$
$\frac{\partial\psi_k}{\partial x}$	$-\frac{1}{h}$	0	0	$-\frac{1}{h}$	0	$\frac{1}{h}$	$\frac{1}{h}$	0

Denote

$$X_N=\left\{u_N(t,x,y)=\sum_{i=1}^{N}\alpha_i(t)\psi_i(x,y),\ \alpha_i(t)\in C^1[0,T],\ \psi_i\in S_n;\ i=1,2,\ldots,N\right\}.$$

To find the weak solution to the problem (1)–(3), we use the Galyorkin method

$$\sum_{i=1}^{N}\int_{\Omega}\alpha_i'(t)\nabla\psi_i\nabla\psi_j\,dx\,dy+\sum_{i=1}^{N}\int_{\Omega}\alpha_i(t)\left(\frac{\partial\psi_i}{\partial x}\cdot\frac{\psi_j}{\partial x}-\frac{\partial\psi_i}{\partial y}\cdot\frac{\psi_j}{\partial y}\right)dx\,dy=$$
$$=\int_{\Omega}f(t,x,y)\psi_i(x,y)\,dx\,dy\,,\quad j=1,2,\ldots,N\,,$$

which is equivalent to

$$\sum_{i=1}^{N}\alpha_i'(t)\left[\psi_i,\psi_j\right]+\sum_{i=1}^{N}\alpha_i(t)\int_{\Omega}\left(\frac{\partial\psi_i}{\partial x}\cdot\frac{\psi_j}{\partial x}-\frac{\partial\psi_i}{\partial y}\cdot\frac{\psi_j}{\partial y}\right)dx\,dy=$$
$$=\int_{\Omega}f(t,x,y)\psi_i(x,y)\,dx\,dy,\qquad j=1,2,\ldots,N. \tag{4}$$

We can rewrite the equality (4) in the matrix form

$$\beta_N'(t)+M_N\beta_N(t)=F_N\,, \tag{5}$$

where $\beta_N=(\alpha_1,\alpha_2,\ldots,\alpha_N)$, $F_N=\left(\int_{\Omega}f\psi_1(x,y)\,dx\,dy,\ldots,\int_{\Omega}f\psi_N(x,y)\,dx\,dy\right)$,

$M_N=\left(\int_{\Omega}\left(\frac{\partial\psi_i}{\partial x}\cdot\frac{\psi_j}{\partial x}-\frac{\partial\psi_i}{\partial y}\cdot\frac{\psi_j}{\partial y}\right)dx\,dy\right)_{i,j=1}^{N}$. It is easy to check that the matrix M_N has the form $M_N=\begin{pmatrix} A & E & 0 & 0 & \cdots & 0 & 0 & 0\\ E & A & E & 0 & \cdots & 0 & 0 & 0\\ 0 & E & A & E & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & E & A & E\\ 0 & 0 & 0 & 0 & \cdots & 0 & E & A\end{pmatrix}$, where E is a unit matrix, and A is the following matrix of order $(n-1)\times(n-1)$:

$$A=\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0\\ -1 & 0 & -1 & \cdots & 0 & 0\\ & -1 & 0 & \cdots & 0 & 0\\ & & -1 & \cdots & -1 & 0\\ & 0 & & \cdots & 0 & -1\\ & & & & -1 & 0\end{pmatrix}.$$

Let us denote by $\alpha_i(t)$ (j=1,2,…,N) the solution to the system of differential equations (5) with conditions

$$\alpha_i(0) = c_i \qquad (i = 1,2,\ldots,N), \tag{6}$$

where c_i are the expansion coefficients of the function $u_0(x,y)$ with respect to basis $\psi_i(x,y)$. Thus, we obtain the following sequence of the functions

$$u_N^*(t,x,y) = \sum_{i=1}^{N} \alpha_i'(t)\psi_i(x,y).$$

The sequence $\{u_N^*(t,x,y)\}_{N=1}^{\infty}$ converges in norm $L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ to the weak solution of the problem (1)–(3) (see Theorem 2 in [6]).

3. To find the numerical solution of the system (5), we use the θ - method (see [7, 8]). Suppose we partition $[0,T]$ into equal parts with step Δt. Denote $\beta_N^k = \beta_N(k\Delta t) = (\alpha_1(k\Delta t),\ldots,\alpha_N(k\Delta t))$. Now we replace the system (5) by the following difference system

$$\frac{\beta_N^{k+1} - \beta_N^k}{\Delta t} + M_N(\theta\beta_N^{k+1} + (1-\theta)\beta_N^k) = \theta F_N^{k+1} + (1-\theta)F_N^k, \tag{7}$$

where $F_N^k = F_N(k\Delta t)$, $0 \le \theta \le 1$.

For every k we get the linear system of equations. We choose the parameter θ such that the matrix $K = \dfrac{E}{\Delta t} + \theta M_N$ is positive $\left(\theta < \min\left\{1; \dfrac{1}{\Delta t\|M_N\|}\right\}\right)$. Then we may represent the system of equations (7) in the following form (see [7])

$$\begin{cases} H^T Y = \left[\dfrac{1}{\Delta t} - (1-\theta)M_N\right]\beta_N^k + \theta F_N^{k+1} + \theta F_N^k \\ H\beta_N^{k+1} = Y \end{cases}, \tag{8}$$

where $K = H^T H$. Denote $u_N^{k+} = \sum_{i=1}^{N} \alpha_i(k\Delta t)\psi_i(x,y)$. It is easy to verify that

$$\left\|u_N^* - u_N^k\right\|_{L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)} = O(\Delta t^2).$$

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Վերջավոր տարրերի մեթոդը մոդելային պսևդոպարաբոլական հավասարման համար

Աշխատանքում ուսումնասիրվում է մոդելային պսևդոպարաբոլական հավասարման համար սկզբնական-եզրային խնդրի լուծման մոտավոր կառուցումը վերջավոր տարրերի մեթոդով:

Ապացուցվում է, որ այդ մեթոդով կառուցված հաջորդականությունը զուգամիտում է ճշգրիտ լուծմանը: Ստացվել է սխալի գնահատականը:

Метод финитных элементов для модельного псевдопараболического уравнения

В работе методом финитных элементов исследуется приблизительное построение начально-краевой задачи для модельного псевдопараболического уравнения.

Доказывается, что последовательность, построенная таким методом, сходится к точному решению. Получена оценка ошибки