PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2009, № 3, p. 32–36

Mathematics

ON MULTIPLE INTERSECTION POINT OF HOMOGENEOUS CURVES

G. S. AVAGYAN*

Chair of Numerical Analysis and Mathematical Modeling, YSU

In the present paper we describe the polynomial space, corresponding to the multiple intersection point of two curves that are given as products of correspondingly m and n different lines, passing through that point.

The main result gives the basis of that space, using polynomials, which correspond to the derivatives with respect to the directions of those lines.

***Keywords*:** algebraic curves, intersection multiplicity.

In the present paper, if not mentioned otherwise, by n-degree polynomial we understand an n-degree bivariate polynomial. Let us denote by $\prod$ the space of all polynomials; by $\prod_n$ and $\prod_n^0$ the n-degree and n-degree homogeneous polynomial spaces respectively.

Given two curves C_1 and C_2, defined by equations $P(x,y)=0$ and $Q(x,y)=0$ respectively, where $P,Q\in\prod$. Let (x_0,y_0) be an intersection point of those curves: $(x_0,y_0)\in C_1\cap C_2$. Define the following space: $M=M_{(x_0,y_0)}\subset\prod$ (see [1]),

1. $r\in M$, $r\left(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial y}\right)P(x_0,y_0)=0$, $r\left(\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial y}\right)Q(x_0,y_0)=0$,
2. $r\in M$, then $\dfrac{\partial}{\partial x}r\in M$, $\dfrac{\partial}{\partial y}r\in M$.

The space $M_{(x_0,y_0)}$ is called *multiplicity of intersection* of curves C_1 and C_2 at point (x_0,y_0), and the dimension of that space is called the *number of intersections* or *arithmetical multiplicity.*

Now we'll describe the space $M_{(x_0,y_0)}$ for the case, when C_1 and C_2 are given as products of correspondingly m and n lines, passing through the point (x_0,y_0). Without loss of generality we assume that $(x_0,y_0)=(0,0)$, since we always can come to it by choosing a convenient coordinate system. It is clear that in this case products of these lines determines a bivariate homogeneous polynomial. We also know the opposite: each homogeneous polynomial of two variables is a product of lines, passing through the point $(0,0)$ (see [2]). So we are

* E-mail: grigoravagyan@yahoo.com

going to describe the polynomial space, corresponding to the multiplicity of intersection at a point of two homogeneous curves, that pass through that point and have no common factors.

Denote $M = M_{(x_0,y_0)}$. To construct a basis of M we'll need the following auxilliary propositions.

Proposition 1. Let $l(x,y) = \alpha x + \beta y$, $p(x,y) = \sum_{i+j=n} a_{ij} x^i y^j$, then $l^n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p(x,y) = n!\, p(\alpha,\beta)$.

Proof. Using the formula $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} a^{n-i} b^i$, we get $l^n(x,y) = \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} \alpha^{n-i} \beta^i x^{n-i} y^i$. Note that if $p(x,y) = \sum_{i+j=n} a_{ij} x^i y^j$ is a homogeneous polynomial and $r(x,y) = x^s y^t$ for $s+t=n$, then $r\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p(x,y) = s!\,t!\,a_{st}$. So we have

$$l^n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p(x,y) = \sum_{i+j=n} a_{ij}\, i!\, j! \frac{n!}{(n-i)!\,i!} \alpha^i \beta^j = n! \sum_{i+j=n} a_{ij} \alpha^i \beta^j = n!\, p(\alpha,\beta).$$

The statement is thus proved.

Particularly for the case $p(x,y) = l_1 l_2 \cdots l_n$, where $l_i = a_i x + b_i y$, $i = 1,2,\ldots,n$, if the point (α,β) belongs to any of lines l_i, then $l^n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) p(x,y) = 0$. In general, $l^n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(l_1 l_2 \cdots l_n) = 0$, iff l has the same direction as one of the lines $l_1, l_2, \ldots, l_n$.

Proposition 2. Given $\{l_i\}_{i=1}^{n+1}$ different lines, passing through the point $(0,0)$, then the following polynomials are independent $l_1^k, l_2^k, \ldots, l_{n+1}^k$, $k \geq n$.

Proof. Obviously, the proposition may be proved only for the case $k = n$. Let $l_j = y + k_j x$, then $l_j^n = \sum_{i=0}^{n} \binom{n}{i} k_j^i x^i y^{n-i}$, $j = 1,2,\ldots,n+1$. Generally, independence of polynomials is equivalent to the independence of vectors, whose components are the polinomials' coefficients. Thus, the above-mentioned polynomials are independent, if the following determinant is non-zero:

$$\begin{vmatrix} 1 & \binom{n}{1} k_1 & \binom{n}{2} k_1^2 & \cdots & \binom{n}{n} k_1^n \\ 1 & \binom{n}{1} k_2 & \binom{n}{2} k_2^2 & \cdots & \binom{n}{n} k_2^n \\ . & . & . & . & . \\ 1 & \binom{n}{1} k_{n+1} & \binom{n}{2} k_{n+1}^2 & \cdots & \binom{n}{n} k_{n+1}^n \end{vmatrix}.$$

The latter is non-zero, if the following Vandermonde determinant is non-zero:

$$\begin{vmatrix} 1 & k_1 & k_1^2 & \cdots & k_1^n \\ 1 & k_2 & k_2^2 & \cdots & k_2^n \\ . & . & . & . & . \\ 1 & k_{n+1} & k_{n+1}^2 & \cdots & k_{n+1}^n \end{vmatrix}.$$

But the Vandermonde determinant is non-zero, since all k_i's are different. The statement is thus proved.

Now, using the Propositions 1 and 2, we can prove the following theorem.

Theorem. Let the curve C_1 be given as a product of different lines $l_1, l_2, ..., l_m$, passing through the point $(0,0)$, $C_1 : P(x,y) = l_1 l_2 \cdots l_m = 0$, and the curve C_2 be given as a product of the lines $\tilde{l}_1, \tilde{l}_2, ..., \tilde{l}_n$ (passing through the point $(0,0)$), different from each other and different from all l_i's, $C_2 : Q(x,y) = \tilde{l}_1 \tilde{l}_2 \cdots \tilde{l}_n = 0$. Given $n \geq m$, L_i and $\tilde{L}_i$'s are the polynomials corresponding to the derivatives by directions of the lines l_i and $\tilde{l}_i$ respectively. Then the following mn polynomials form the basis of M :

$$\begin{array}{cccccccc|ccccc}
1 & L_1 & L_1^2 & L_1^3 & \cdots & L_1^{m-1} & \cdots & L_1^{n-1} & & & & & \\
 & L_2 & L_2^2 & L_2^3 & \cdots & L_2^{m-1} & \cdots & L_2^{n-1} & \{L_2^n, L_1^n\} & & & & \\
 & & L_3^2 & L_3^3 & \cdots & L_3^{m-1} & \cdots & L_3^{n-1} & \{L_3^n, L_1^n\} & \{L_3^{n+1}, L_1^{n+1}, L_2^{n+1}\} & & & \\
 & & & L_4^3 & \cdots & L_4^{m-1} & \cdots & L_4^{n-1} & \{L_4^n, L_1^n\} & \{L_4^{n+1}, L_1^{n+1}, L_2^{n+1}\} & \{L_4^{n+2}, L_1^{n+2}, L_2^{n+2}, L_3^{n+2}\} & & \\
 & & & & . & . & . & . & . & . & . & \cdots & \\
 & & & & & L_m^{m-1} & \cdots & L_m^{n-1} & \{L_m^n, L_1^n\} & \{L_m^{n+1}, L_1^{n+1}, L_2^{n+1}\} & \{L_4^{n+2}, L_1^{n+2}, L_2^{n+2}, L_3^{n+2}\} & \cdots & \{L_m^{n+m-2}, L_1^{n+m-2}, \ldots, L_{m-1}^{n+m-2}\}.
\end{array}$$

Here $\{L_1, ..., L_k\}$ denotes a linear combination of polynomials $L_1, ..., L_k$, whose coefficients have to be found during the proof.

Proof. Since in the given scheme polynomials, belonging to the different columns, have different degrees, then in order to show the independence of all polynomials one has to show only the independence of polynomials in each column. Independence of polynomials in the left side of the scheme follows from the Proposition 2, and from the fact that lines l_i, $i = 1, 2, ..., m$, are different.

The cases $L_i^j \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) P(x,y) \Big|_{(0,0)} = 0 \;\; \forall\ i = 1, 2, ..., m$, $\forall\ j = 0, 1, 2, ...\ j < m$ and $j > m$ are obvious, and the case $j = m$ follows from the Proposition 1. The same property holds for all derivatives of the polynomial L_i^j, because $\frac{\partial}{\partial x} L^k(x,y) = kaL^{k-1}(x,y)$ and $\frac{\partial}{\partial y} L^k(x,y) = kbL^{k-1}(x,y)$, where $L(x,y) = ax + by$. So, all polynomials in the scheme and their derivatives turn the polynomial $P(x,y) = l_1 l_2 \cdots l_m$ to zero at point $(0,0)$.

It is obvious that all polynomials in the left side of the scheme, as well as their derivatives, turn the polynomial $Q(x,y)=\tilde{l}_1\tilde{l}_2\cdots\tilde{l}_n$ to zero at the point $(0,0)$, since their degrees are all less than n. Now let us show that one can choose the coefficients of linear combinations, so that all the polynomials at the right side of the scheme and their derivatives turn the polynomial $Q(x,y)$ to zero. It is easy to see that the latter proposition is the same as to show, that the coefficients of linear combinations may be taken so that the ($r-1$)-th derivatives $\left(\frac{\partial^{i+j}}{\partial x^i \partial y^j}, i+j=r-1\right)$ of polynomials of each r-th column zeroing the polynomial Q at the point $(0,0)$. Let us consider any polynomial from the r-th column: $cL_h^{n+r-1}+a_1L_1^{n+r-1}+...+a_rL_r^{n+r-1}$, $h>r$. Obviously, under the condition that all ($r-1$)-th derivatives of this polynomial turn $Q(x,y)$ to zero at point $(0,0)$, we come to a linear system of r independent equations with $r+1$ variables that has exactly one (with respect to multiplication by a number) non-zero solution.

To show the independence one has to show that the coefficient c in that solution is not zero. Let us consider the equation system, which determines $c, a_1, a_2, ..., a_r$:

$$\left.\left(\frac{\partial^{i+j}}{\partial x^i \partial y^j}F\right)Q(x,y)\right|_{(0,0)}=0, \qquad i+j=r-1,$$
$$F=cL_h^{n+r-1}+a_1L_1^{n+r-1}+...+a_rL_r^{n+r-1}. \tag{*}$$

Let $L_s=y+k_sx$, then we have

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j}F=\left(ck_h^iL_h^n+a_1k_1^iL_1^n+...+a_rk_r^iL_r^n\right)\left(n+r-1\right)\cdot...\cdot(n+1),\ i=0,1,...,r-1.$$

Since $L_i^nQ(x,y)=n!Q(\alpha,\beta)\neq 0$ (according to the Proposition 1 and the condition that l_i is not a factor of Q) and all k_i's are different, then the determinant

$$D=\begin{vmatrix} 1 & \cdots & 1 \\ a_1k_1L_1^nQ & \cdots & a_rk_rL_r^nQ \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ a_1k_1^{r-1}L_1^nQ & \cdots & a_rk_r^{r-1}L_r^nQ \end{vmatrix}$$

is not zero. $D\neq 0$ implies that c in the solution of the system (*) is not zero, otherwise, we would come to the contradiction that a system of homogeneous linear equations with non-singular determinant has a non-zero solution. Now, using this result and the Proposition 2, it is easy to see that the polynomials of each column in the right side of the scheme are independent.

So we get mn independent polynomials belonging to M. Since C_1 and C_2 have no common factors (no common lines), then according to Bezout's Theorem (see [1]), those curves have mn intersection points including multiple ones. But since the only intersection point of these lines is $(0,0)$, then the arithmetical multiplicity of intersection at $(0,0)$ is mn, and, therefore, $\dim M = mn$. Thus the polynomials given in the scheme form a basis in M.

The Theorem is proved.

From the proof it is clear that the condition $m \le n$ is not significant, and we could construct the scheme, using the lines $\tilde{l}_i$ as well. Then from the Theorem it follows that the linear spans of polynomials in the both schemes coincide.

Received 06.03.2009

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Համասեռ կորերի հատման կետի պատիկության մասին

Աշխատանքում նկարագրվում է կետի պատիկությանը համապա-տասխանող բազմանդամային տարածությունը այդ կետով անցնող ուղիղների արտադրյալով որոշվող երկու կորերի համար: Աշխատանքի հիմնական արդյունքը տալիս է այդ տարածության բազիսն այն բազմանդամների միջոցով, որոնք համապատասխանում են ըստ նշված ուղղությունների ածանցյալներին:

О кратности точки пересечения однородных кривых

В работе описывается полиномиальное пространство, соответствующее кратности точки пересечения двух кривых, которые определяются произведением прямых, проходящих через эту точку. Основной результат работы дает базис этого пространства через полиномы, соответствующие производным по указанным направлениям.