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Mathematics

ON $\{2,3\}$-HYPERIDENTITIES IN INVERTIBLE $\{2,3\}$-ALGEBRAS WITH A BINARY GROUP OPERATION

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In the present paper the invertible $\{2,3\}$-algebras with a binary group operation and with balanced first sort $\{2,3\}$-hyperidentities having length 4 are characterized.

Keywords**:** quasigroup, ternary quasigroup, hyperidentity, invertible algebra.

Introduction. Let Q be a nonempty set and n be a nonnegative integer. We define Q^n as the direct power of set Q, i.e. Q^n is the set of n-tuples of elements from Q. The mapping $A:Q^n \to Q$ is called n-ary operation on the set Q, and the number n is the arity of the operation A. The pair $Q(A)$, where A is a n-ary operation, is called n-quasigroup or n-ary quasigroup [1, 2], if any set of n elements out of $x_1, x_2, \ldots, x_n, x_{n+1}$ uniquely defines the n+1-st one in the equality $A(x_1, x_2, \ldots, x_n) = x_{n+1}$. In other words, $Q(A)$ is a n-quasigroup, if the equation $A(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = b$ has a unique solution for any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n,\ b \in Q$ and for any $i = 1, 2, \ldots, n$. In this case the operation A is called a quasigroup operation. For $n=1$ the operation A is a bijection, and for $n=2$ the n-quasigroup $Q(A)$ is called a binary quasigroup or simply a quasigroup, and for $n=3$ the n-quasigroup $Q(A)$ is called a ternary quasigroup.

Let A be a quasigroup operation of arity n, and let $A(x_1, \ldots, x_n) = y$. If we replace the elements $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ by fixed elements $a_1, a_2, \ldots, a_m \in Q$, respecttively, then $A(x_1, \ldots, x_n)$ takes the form $A(x_1, \ldots, x_{k_1-1}, a_1, x_{k_1+1}, \ldots, x_{k_2-1}, a_2, \ldots)$, and we come to the new operation $B(x_1, \ldots, x_{k_1-1}, x_{k_1+1}, \ldots, x_{k_2-1}, \ldots, x_n)$ of arity $n-m$. Obviously, B is a quasigroup operation, which is called the retract of A.

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The following second order formula [3] is called a hyperidentity [4, 5]:

$$\forall x_1,...,x_k \forall X_1,...,X_m (W_1 = W_2),$$

where $X_1,...,X_m$ are functional variables, and $x_1,...,x_k$ are subject variables in words (terms) W_1, W_2. The number m is called a rank of a hyperidentity. Usually a hyperidentity is specified without universal quantifier prefix: W_1=W_2. If a rank m>1, the hyperidentity is called nontrivial, and if arities are $|X_1|=n_1,...,\ |X_m|=n_m$, the hyperidentity W_1=W_2 is called an $\{n_1,...,n_m\}$-hyperidentity. The satisfiability of a hyperidentity is defined in accordance with its quantifier prefix.

A hyperidentity is called *balanced*, if each subject variable of the hyperidentity occurs in the both parts of the latter one, whereas only once. A balanced hyperidentity is called a *first sort* hyperidentity, if the subject variables in the left and right parts of the equality are ordered identically. The number of subject variables in a balanced hyperidentity is called the *length* of this hyperidentity. The classification of associative hyperidentities and the criteria of their satisfiability in q- and e-algebras are given in [4–6]. For solutions of analogical problems for ternary associativity hyperidentities in invertible algebras see in [4] (also see [2]).

In the present paper the criteria of satisfiability of first sort balanced $\{2,3\}$-hyperidentities of length 4 in invertible $\{2,3\}$-algebras with binary group operation are investigated.

§ 1. Preliminary Results. We call the algebra $Q(\Sigma)$ with binary and ternary operations a $\{2,3\}$-*algebra*. The $\{2,3\}$-algebra is *nontrivial*, if the sets of its binary and ternary operations are not one-element. The algebra $Q(\Sigma)$ is called *invertible*, if $Q(A)$ is a quasigroup (for some arity) for any operation $A \in \Sigma$.

Theorem 1 [7].

1. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x,\ y,\ z),u) = (x,(y,z,u)),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$X(Y(x,y,z),u) = X(x,Y(y,z,u)). \tag{1}$$

2. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x,\ y),u,v) = (x,(y,\ u),v),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$Y(X(x,y),u,v) = Y(x,X(y,u),v). \tag{2}$$

3. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x,\ y),u,v) = (x,y,(u,\ v)),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$Y(X(x,y),u,v) = Y(x,y,X(u,v)). \tag{3}$$

4. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x, y, z), u) = ((x, y), z, u),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$X(Y(x, y, z), u) = Y(X(x, y), z, u). \quad (4)$$

5. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x, y, z), u) = (x, (y, z), u),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$X(Y(x, y, z), u) = Y(x, X(y, z), u). \quad (5)$$

6. If the $\{2,3\}$-hyperidentity, given by the equality

$$((x, y, z), u) = (x, y, (z, u)),$$

is satisfied in an invertible nontrivial $\{2,3\}$-algebra, then every functional variable is repeated in it at least twice. Hence, such hyperidentity has the unique form:

$$X(Y(x, y, z), u) = Y(x, y, X(z, u)). \quad (6)$$

§ 2. Satisfiability Criterion for Hyperidentities (1)–(6).

Theorem 2.1. Suppose that an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has a binary operation $(\cdot) \in \Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (1) is satisfied in the algebra $Q(\Sigma)$, iff each ternary operation $A_i \in \Sigma$ is defined by the rule $A_i(x, y, z) = x \cdot y \cdot z \cdot t_i$, where $t_i \in Z(Q)$ (which is the center of the group $Q(\cdot)$), and each binary operation $B_j \in \Sigma$ is defined by the rule $B_j(x, y) = \alpha_j(x \cdot y)$, where $\alpha_j : Q \to Q$ is a bijection.

Proof: The proof of sufficiency is established by direct checking:

$$\alpha_j(x \cdot y \cdot z \cdot t_i \cdot u) = \alpha_j(x \cdot y \cdot z \cdot u \cdot t_i).$$

Let's prove its necessity.

Let the hyperidentity (1) be satisfied in the given algebra $Q(\Sigma)$. If we substitute $X = (\cdot), Y = A_i$ in (1), then we get $A_i(x, y, z) \cdot u = x \cdot A_i(y, z, u)$.

If here $u = e$, which is the unit of the group $Q(\cdot)$, then

$$A_i(x, y, z) = x \cdot A_i(y, z, e) = x \cdot \lambda_i(y, z),$$

where λ_i is a quasigroup operation, and $\lambda_i(y, z) = A_i(y, z, e)$.

Coming back to the equality (1), we get $x \cdot \lambda_i(y, z) \cdot u = x \cdot y \cdot \lambda_i(z, u)$, whence we get $\lambda_i(y, z) \cdot u = y \cdot \lambda_i(z, u)$, and at $u = e$ we have $\lambda_i(y, z) = y \cdot \lambda_i(z, e) = y \cdot \mu_i(z)$, where $\mu_i : Q \to Q$ is a bijection, i.e.

$$A_i(x, y, z) = x \cdot \lambda_i(y, z) = x \cdot y \cdot \mu_i(z).$$

Again, taking into account (1), we get $x \cdot y \cdot \mu_i(z) \cdot u = x \cdot y \cdot z \cdot \mu_i(u)$, $\mu_i(z) \cdot u = z \cdot \mu_i(u)$, and at $u = e$ we have $\mu_i(z) = z \cdot \mu_i(e) = z \cdot t_i$, where $t_i = \mu_i(e) \in Q$, i.e. $A_i(x, y, z) = x \cdot y \cdot z \cdot t_i$. Now, taking into account the previous equality, we get $z \cdot t_i \cdot u = z \cdot u \cdot t_i$, i.e. $t_i \in Z(Q)$ is the center of the group $Q(\cdot)$.

Thus, each ternary operation $A_i \in \Sigma$ has the following form: $A_i(x,y,z) = x\cdot y\cdot z\cdot t_i$, where $t_i \in Z(Q)$.

Further, if B_j is any binary operation from Σ, and we substitute $X = B_j, Y = A_i$ in (1), then we get $B_j(x\cdot y\cdot z\cdot t_i, u) = B_j(x, y\cdot z\cdot u\cdot t_i)$, and at $x = e,\ z = t_i^{-1}$, we have $B_j(y,u) = B_j(e, y\cdot t_i^{-1}\cdot u\cdot t_i) = B_j(e, y\cdot u) = \alpha_j(y\cdot u)$, where $\alpha_j : Q \to Q$ is a bijection, and $\alpha_j(x) = B_j(e,x)$. □

Theorem 2.2. Suppose an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has the binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (2) is satisfied in the algebra $Q(\Sigma)$, iff each ternary operation $A_i \in \Sigma$ is defined by the rule $A_i(x,y,z) = \lambda_i(x\cdot y, z)$, where $\lambda_i : Q^2 \to Q$ is a quasigroup operation and each binary operation $B_j \in \Sigma$ is defined by the rule $B_j(x,y) = x\cdot y\cdot t_i$, where $t_j \in Z(Q)$ (the center of the group $Q(\cdot)$).

Proof: The proof of sufficiency is established by direct checking:

$$\lambda_i(x\cdot y\cdot t_i\cdot u, v) = \lambda_i(x\cdot y\cdot u\cdot t_i, v).$$

Now let's prove its necessity.

Let the hyperidentity (2) be satisfied in the given algebra $Q(\Sigma)$. If we substitute $X = (\cdot), Y = A_i$ in (2), then we have $A_i(x\cdot y, u, v) = A_i(x, y\cdot u, v)$.

If $x = e$, then $A_i(y,u,v) = A_i(e, y\cdot u, v) = \lambda_i(y\cdot u, v)$, where λ_i is a quasigroup operation, and $\lambda_i(x,y) = A_i(e,x,y)$. Hence, from (2) at $X = B_j, Y = A_i$ we get $\lambda_i(B_j(x,y)\cdot u, v) = \lambda_i(x\cdot B_j(y,u), v)$, whence after reduction by v we have $B_j(x,y)\cdot u = x\cdot B_j(y,u)$, and at $u = e$ we get

$$B_j(x,y) = x\cdot B_j(y,e) = x\cdot \mu_j(y),$$

where $\mu_j(y) = B_j(y,e)$ and $x\cdot\mu_j(y)\cdot u = x\cdot y\cdot \mu_j(u)$, i.e. $\mu_j(y)\cdot u = y\cdot\mu_j(u)$, and at $u = e$ we get $\mu_i(y) = y\cdot\mu_j(e) = y\cdot t_j,\ t_j = \mu_j(e)\in Q$. Therefore, $y\cdot t_j\cdot u = y\cdot u\cdot t_j$, i.e. $t_j \in Z(Q)$. □

Theorem 2.3. Suppose an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has the binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (3) is satisfied in the algebra $Q(\Sigma)$, iff each ternary operation $A_i \in \Sigma$ is defined by the rule $A_i(x,y,z) = \mu_i(x\cdot y\cdot z)$, where $\mu_i : Q\to Q$ is a bijection, and each binary operation $B_j \in \Sigma$ is defined by the rule $B_j(x,y) = x\cdot y\cdot t_i$, where $t_j \in Z(Q)$ (which is the center of the group $Q(\cdot)$).

Proof: The proof of sufficiency is established by direct checking:

$$\mu_i(x\cdot y\cdot t_i\cdot u\cdot v) = \mu_i(x\cdot y\cdot u\cdot v\cdot t_i).$$

Let's prove its necessity.

Let the hyperidentity (3) be satisfied in the given algebra $Q(\Sigma)$. If we substitute $X = (\cdot), Y = A_i$ in (3), then we get $A_i(x\cdot y, u, v) = A_i(x, y, u\cdot v)$, whence at $x = e$ we have $A_i(y,u,v) = A_i(e, y, u\cdot v) = \lambda_i(y, u\cdot v)$, where λ_i is a quasigroup operation, and $\lambda_i(x,y)=A_i(e,x,y)$. Hence, $\lambda_i(x\cdot y,u\cdot v)=\lambda_i(x,y\cdot u\cdot v)$, and at $x=e=u$ we get $\lambda_i(y,v)=\lambda_i(e,y\cdot v)=\mu_i(y\cdot v)$, where μ_i is a bijection, and $\mu_i(x)=\lambda_i(e,x)$. Thus, $A_i(x,y,z)=\lambda_i(x,y\cdot z)=\mu_i(x\cdot y\cdot z)$.

Again, from (3) at $X=B_j$ and $Y=A_i$ we get

$$\mu_i(B_j(x,y)\cdot u\cdot v)=\mu_i(x\cdot y\cdot B_j(u,v)).$$

Therefore, $B_j(x,y)\cdot u\cdot v=x\cdot y\cdot B_j(u,v)$, and at $u=v=e$ we have $B_j(x,y)=x\cdot y\cdot B_j(e,e)=x\cdot y\cdot t_j$, and $x\cdot y\cdot t_j\cdot u\cdot v=x\cdot y\cdot u\cdot v\cdot t_j$, i.e. $t_j\in Z(Q)$. □

The following results have similar proofs.

Theorem 2.4. Suppose an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has the binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (4) is satisfied in algebra $Q(\Sigma)$, iff each ternary operation $A_i\in\Sigma$ is defined by the rule

$$A_i(x,y,z)=t_i\cdot x\cdot y\cdot z,\quad t_i\in Q,$$

and each binary operation $B_j\in\Sigma$ is defined by the rule $B_j(x,y)=s_j\cdot x\cdot y$, $s_j\in Q$, where $s_j\cdot t_i=t_i\cdot s_j$.

Theorem 2.5. Suppose an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has the binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (5) is satisfied in the algebra $Q(\Sigma)$, iff each ternary operation $A_i\in\Sigma$ is defined by the rule $A_i(x,y,z)=\theta_i(x)\cdot y\cdot z$, where $\theta_i:Q\to Q$ is a bijection, and each binary operation $B_j\in\Sigma$ is defined by the rule $B_j(x,y)=x\cdot y\cdot \ell_i$, where $\ell_i\in Z(Q)$ (which is the center of the group $Q(\cdot)$).

Theorem 2.6. Suppose an invertible $\{2,3\}$-algebra $Q(\Sigma)$ has the binary operation $(\cdot)\in\Sigma$, such that $Q(\cdot)$ is a group. Then the hyperidentity (6) is satisfied in the algebra $Q(\Sigma)$, iff each ternary operation $A_i\in\Sigma$ is defined by the rule $A_i(x,y,z)=\mu_i(x,y)\cdot u$, here $\mu_i:Q^2\to Q$ is a quasigroup operation and each binary operation $B_j\in\Sigma$ is defined by the rule $B_j(x,y)=x\cdot\varphi_i(y)$, where $\varphi_j:Q\to Q$ is a bijection.

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REFERENCES

1. **Belousov V.D.** *n*-ary Quasigroups. Kishinev: Shtiintsa, 1972 (in Russian).
2. **Usan J.** *n*-groups in the Light of the Neutral Operations. Electronic version, 2006.
3. **Maltsev A.I.** Algebraic Systems. M.: Nauka, 1970 (in Russian).
4. **Movsisyan Yu.M.** Introduction to the Theory of Algebras with Hyperidentities. Yerevan: YSU Press, 1986 (in Russian).
5. **Movsisyan Yu.M.** Hyperidentities and Hypervarieties in Algebras. Yerevan: YSU Press, 1990 (in Russian).
6. **Movsisyan Yu.M.** Russian Math. Surv., 1998, v. 53, № 1, p. 57–108.
7. **Ghumashyan H.E.** Uch. Zapiski EGU, 2008, № 1, p. 141–142 (in Russian).

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Երկտեղ խմբային գործողությամբ {2, 3}-հանրահաշիվներում
{2, 3}-գերնույնությունների մասին

Աշխատանքում բնութագրվում են երկտեղ խմբային գործողությամբ այն {2, 3}-հանրահաշիվները, որոնք բավարարում են 4 երկարությամբ առաջին սեռի հավասարակշռված {2, 3}-գերնույնություններին:

О {2, 3}-сверхтождествах в обратимых {2, 3}-алгебрах
с бинарной групповой операцией

В работе характеризуются обратимые $\{2,3\}$-алгебры с бинарной групповой операцией и с уравновешенными $\{2,3\}$-сверхтождествами первого рода длины 4.