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COMMUNICATIONS

Mathematics

ON SYMMETRIC SUBALGEBRAS IN BANACH ALGEBRA

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Symmetric subalgebras of a complex Banach algebra are studied with the help of von Neiman–Fuglede Theorem. The result is closely related to the problems of approximation theory.

Keywords: symmetric Banach algebra.

Let A be a Banach algebra with unity **1** over the field of complex numbers $\mathbb{C}$ ([1, 2]).

Note that $\mathbb{C}$ is a linear functional $\psi : A \to \mathbb{C}$ called a *state*, if $\|\psi\| = \psi(\mathbf{1}) = 1$. The set $St(A)$ of all states is $\sigma(A^*, A)$ a compact, convex subset of the conjugate space A^*.

A convex compact

$$V(a) = \{\psi(a) \in \mathbb{C} : \psi \in St(A)\}$$

is called (algebraic) *numerical image* of element $a \in A$. Taking into account the fact, that $V(a) = \bigcap_{z \in \mathbb{C}} \Delta(z; \|z\mathbf{1} - a\|)$, where $\Delta(z; \|z\mathbf{1} - a\|) = \{\xi \in \mathbb{C} : |\xi - z| \le \|z\mathbf{1} - a\|\}$, it is easy to see, that $sp(a) \subset V(a)$ holds for any element $a \in A$. Here $sp(a)$ is the spectrum of the element $a \in A$.

Element $\mathbf{h} \in A$ is called "*hermitian*", if $V(\mathbf{h}) \subset \mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. The set $H(A)$ of the all hermitian elements in algebra A is a closed $\mathbb{R}$-linear subset in A.

The condition $\mathbf{h} \in H(A)$ is equivalent to $\|\exp(it\mathbf{h})\| = 1$ for all $t \in \mathbb{R}$. The equality $\|\mathbf{h}\| = \rho(\mathbf{h})$ holds for any $\mathbf{h}$ hermitian element, where $\rho(\mathbf{h})$ is the spectral radius of the element $\mathbf{h}$. The latter equality is equivalent to the classical S.N. Bernstein inequality for the derivative of an exponential type entire function bounded on the real axis (see [3–6]).

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Note that an element $a \in A$ is called "hermitian"-expandable, if it may be represented as $a = \mathbf{h} + i\mathbf{k}$, where $\mathbf{h,k} \in H(A)$. If there is such a representation, then it is unique. The system of all expandable elements of the A-algebra is a $\mathbb{C}$-linear closed subspace in algebra A, which is denoted as $H_{\mathbb{C}}(A)$.

The element $a = \mathbf{h} + i\mathbf{k}$ under condition $[\mathbf{h,k}] = \mathbf{hk} - \mathbf{kh} = 0$ is called a normal element of the algebra A, and the element $a^{+} = \mathbf{h} - i\mathbf{k}$ is called its "conjugate" element.

It is easy to see, that for any normal element $a \in A$ there holds $V(a) = \langle sp(a) \rangle$, where $\langle sp(a) \rangle$ is a convex spectrum shell of the element $a \in A$.

Proposition 1. If $\mathbf{h},\mathbf{k} \in H(A)$, then for $m = 1,2,\dots$ it holds $i^{m} \underbrace{[\mathbf{h},[\mathbf{h},...,[\mathbf{h},\mathbf{k}]...]]}_{m-times} \in H(A)$.

Proof. Let $\varphi \in St(A)$ and $t \in \mathbb{R}$. We consider a functional φ_t, defined by formula

$$\varphi_t(x) = \varphi\left[\exp(it\mathbf{h}) \cdot x \cdot \exp(-it\mathbf{h})\right].$$

It follows from the definition that $\varphi_t \in St(A)$ for any $t \in \mathbb{R}$. It means that for $\mathbf{k} \in H(A)$ we have $\varphi_t(\mathbf{k}) \in \mathbb{R}$. On the other hand, since

$$\exp(it\mathbf{h}) \cdot x \cdot \exp(-it\mathbf{h}) = x + it[\mathbf{h},x] + \frac{(it)^2}{2!}\left[\mathbf{h},[\mathbf{h},x]\right] + \cdots + \frac{(it)^m\left[\mathbf{h},\left[\mathbf{h},\ldots,[\mathbf{h},x]\ldots\right]\right]}{m!} + \cdots,$$

then $\varphi_t(x) = \sum\limits_{m=0}^{\infty} \dfrac{(it)^m \varphi\left(\left[\mathbf{h},\left[\mathbf{h},\ldots,[\mathbf{h},x]\ldots\right]\right]\right)}{m!}$. Therefore,

$\left.\dfrac{d^m \varphi_t(\mathbf{k})}{dt^m}\right|_{t=0} = \varphi\left(i^m\left[\mathbf{h},\left[\mathbf{h},\ldots,[\mathbf{h},\mathbf{k}]\ldots\right]\right]\right) \in \mathbb{R}$, and so $i^m\left[\mathbf{h},\left[\mathbf{h},\ldots,[\mathbf{h},\mathbf{k}]\ldots\right]\right]$ is an hermitian element.

Proposition 2. If $\mathbf{h} \in H(A)$ and $\mathbf{h}^2 \in H_{\mathbb{C}}(A)$, then $\mathbf{h}^2 \in H(A)$.

Proof. Assume $\mathbf{h}^2 = \mathbf{p} + i\mathbf{q}$, where $\mathbf{p}, \mathbf{q} \in H(A)$. But then $\mathbf{h}(\mathbf{p} + i\mathbf{q}) = (\mathbf{p} + i\mathbf{q})\mathbf{h}$, whence it follows that $[\mathbf{h},\mathbf{p}] = i[\mathbf{q},\mathbf{h}]$. According to Proposition 1, $[\mathbf{h},\mathbf{p}]$ and $i[\mathbf{h},\mathbf{p}]$ are hermitian elements, therefore, $[\mathbf{h},\mathbf{p}] = 0$ and analogously $[\mathbf{h},\mathbf{q}] = 0$. It means that $[\mathbf{h},\mathbf{q}] = 0$ and the element $\mathbf{h}^2$ is a normal element. Taking into account that $V(\mathbf{h}^2) = \langle sp(\mathbf{h}^2) \rangle$, we have $V(\mathbf{h}^2) \subset \mathbb{R}$ and hence $\mathbf{h}^2 \in H(A)$.

Note that for a hermitian element $\mathbf{h}$, the element $\mathbf{h}^2$ may be non-expandable.

A set $S \subset H_{\mathbb{C}}(A)$ is called a *normal* set, if S is a commutative subset in algebra A, and each element in S is a normal element of the algebra A.

Theorem. Let B be a maximal normal subset in algebra A. Then B is a closed commutative and symmetric with respect to "conjugation" subalgebra in algebra A.

Proof. First of all we show, that if $a \in A$ is a normal element, and $[a,b]=0$ for any $b \in B$, then $a, a^+ \in B$. Indeed, since $[a,b]=0$, according to the von Neiman–Fuglede Generalized Theorem [7, 8], we have $a^+b = ba^+$. But then the set $\{a, a^+\} \cup B$ is a normal set, and since B is a maximal normal set, then $a, a^+ \in B$. Now let us show that B is closed. Let $a \in B$. Then there exists a sequence of elements $\{a_n\} \subset B$ such that $a_n \to a$ ($n \to \infty$). Since $[a_n, b]=0$, then according to the above-mentioned von Neiman-Fuglede Theorem, conditions $\left[a_n^+, b\right]=0$, $\left[a_n, b^+\right]=0$ hold for each $b \in B$. If $a_n = \mathbf{h}_n + i\mathbf{k}_n$, then $a_n - a_m = (\mathbf{h}_n - \mathbf{h}_m) + i(\mathbf{k}_n - \mathbf{k}_m)$. Since $\|\mathbf{h}_n - \mathbf{h}_m\| \le 2\|a_n - a_m\|$, $\|\mathbf{k}_n - \mathbf{k}_m\| \le 2\|a_n - a_m\|$, then due to the completeness of $H(A)$ there exist limits $\mathbf{h} = \lim_n \mathbf{h}_n$, $\mathbf{k} \lim_n \mathbf{k}_n$ and $\mathbf{h}, \mathbf{k} \in H(A)$. Since $[\mathbf{h}_n, \mathbf{k}_n]=0$ for each natural n, then $[\mathbf{h}, \mathbf{k}]=0$ as $n \to \infty$. Due to the fact that $a_n = \mathbf{h}_n + i\mathbf{k}_n$ and $\mathbf{h}_n \to \mathbf{h}$, $\mathbf{k}_n \to \mathbf{k}$, we have $a_n \to a = \mathbf{h} + i\mathbf{k}$, where $a \in A$ is a normal element. Since $[a_n, b]=0$, $\left[a_n^+, b\right]=0$, $\left[a_n, b^+\right]=0$ for any $n \in \mathbb{N}$, then $[a,b]=0$, $\left[a^+, b\right]=0$, $\left[a, b^+\right]=0$ as $n \to \infty$. Since B is the maximal subset, we have $a, a^+ \in B$.

It is clear that if $a, b \in B$ then since $B = Z(Z(B))$, $a+b$ and ab belong to B. Thus, B is a closed, commutative and symmetric with respect to “conjugation” subalgebra in algebra A.

Corollary 4. Let B be a maximal normal subset in algebra A. Then for any $a \in B$ holds $sp_A(a) = sp_B(a)$.

The proof follows from Theorem.

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REFERENCES

1. **Rudin W.** Functional Analysis. M.: Mir, 1975 (in Russian).
2. **Naimark M.A.** Normed Rings. M.: Nauka, 1968 (in Russian).
3. **Akhiezer N.I.** Lectures on the Approximation Theory. M.: Nauka, 1965 (in Russian).
4. **Asatryan H.A., Khachatryan I.G., Karakhanyan M.I., Kamalyan A.H.** Banach Algebras and Spectral Theory. Yerevan State University Publisher, 2008, 251 p. (in Russian).
5. **Gorin E.A.** Vestnik Kharkov. Univ., Series Prikl. Math. i Mekh., 1980, v. 45, № 205, p. 77–105.
6. **Gorin E.A.** Functional Analysis and Its Applications, 2005, v. 39, № 4, p. 14–31.
7. **Karakhanyan M.I.** Functional Analysis and Its Applications, 2005, v. 39, № 4, p. 80–83 (in Russian).
8. **Karakhanyan M.I.** Izv. NAN Armenii. Matematika, 2007, v. 42, № 3, p. 49–54.

Բանախյան հանրահաշվի սիմետրիկ ենթահանրահաշիվների մասին

Աշխատանքում ուսումնասիրվում են կոմպլեքս բանախյան հանրա-հաշվի համար սիմետրիկ ենթահանրահաշիվները ֆոն Նեյման–Ֆուգլեդեի թեորեմի օգնությամբ: Այս արդյունքը սերտորեն կապված է մոտարկման տեսության հարցերի հետ:

О симметричных подалгебрах в банаховой алгебре

В работе исследуются симметричные подалгебры в банаховой алгебре с помощью обобщенной теоремы фон Неймана–Фугледе. Этот результат тесно связан с вопросами теории аппроксимации.