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Mathematics

WEIERSTRASS AND BLASCHKE TYPE FUNCTIONS

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In the present paper the Weierstrass multipliers are generalized and the theorem of convergence of corresponding infinite products is proved. Representations of Blaschke type functions are established through Weierstrass type functions and Blaschke function. A way construction of new Blaschke type products is shown, and a method of proof of their convergence is developed. Some relations between Blaschke type products are established.

***Keywords*:** infinite products, Weierstrass products, Blaschke and Djrbashyan type products.

1. The Weierstrass and Blaschke type infinite products play an important role in the theory of classes factorization of functions, meromorphic in various domains. In [1] it is shown, that quite wide classes of such products have identical structure. Let $D=\{z:|z|<1\}$ be a unit disc, and $G=\{z:\operatorname{Im}z<0\}$ be the bottom half-plane. Denote

$$b_0^{(1)}(z,\varsigma)=\frac{\varsigma-z}{1-\bar{\varsigma}z}\bar{\varsigma}\,,\quad z,\varsigma\in D\,;$$

$$b_0^{(2)}(z,\varsigma)=b_0^{(3)}(z,\varsigma)=\frac{z-\varsigma}{z-\bar{\varsigma}},\quad z,\varsigma\in G\,.$$

In [1] it is established, that for elementary factors $b_\alpha^{(l)}=b_\alpha^{(l)}(z,\varsigma)$ $(l=1,2,3)$ of Blaschke–Djrbashyan type infinite products, where $\alpha\in(-1,+\infty)$, the following integral representations are true:

$$b_\alpha^{(1)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(1)}}(1-t)^\alpha\frac{dt}{t}\right\},\qquad z,\varsigma\in D\,, \tag{1}$$

$$b_\alpha^{(2)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(2)}}\left(\frac{1-t}{1+t}\right)^\alpha\frac{dt}{t}\right\},\qquad z,\varsigma\in D\,, \tag{2}$$

$$b_\alpha^{(3)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(3)}}(1-t)^\alpha\frac{dt}{t}\right\},\qquad z,\varsigma\in G\,, \tag{3}$$

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where integrals are taken along any contours laying inside the disc D and passing through points 1 and $b_0^{(l)} = b_0^{(l)}(z,\varsigma)$ $(l=1,2,3)$, and not passing through the beginning of coordinates at $z \neq \varsigma$.

In [2], [3] it is proved, that if the sequence of complex numbers $Z = \{z_n\}_1^\infty \subset D$ satisfies the condition

$$\sum_{n=1}^{\infty} \left(1-|z_n|\right)^{1+\alpha} < +\infty, \qquad \alpha \in (-1,+\infty), \tag{4}$$

then the infinite product

$$B_\alpha^{(1)}(z,\{z_n\}) = \prod_{n=1}^{\infty} b_\alpha^{(1)}(z,z_n) \tag{5}$$

converges absolutely and uniformly inside D and represents an analytical function with zeros Z.

In [4], [5] and [6] it is proved, that if the sequence of complex numbers $Z = \{z_n\}_1^\infty \subset G$ satisfies the condition

$$\sum_{n=1}^{\infty} (\operatorname{Im} z_n)^{1+\alpha} < +\infty, \qquad \alpha \in (-1,+\infty), \tag{6}$$

then the infinite product

$$B_\alpha^{(l)}(z,\{z_n\}) = \prod_{n=1}^{\infty} b_\alpha^{(l)}(z,z_n) \quad (l=2,3) \tag{7}$$

converges absolutely and uniformly inside G and represents an analytical function with zeros Z. In [1] the following way of construction of new Blaschke–Djrbashyan type functions is shown. Denote by Φ a class of analytic functions inside the unit disc D, for which $\varphi(0)=1$, and the integrals $\int_1^z \frac{\varphi(t)}{t}dt$, $0<|z|<1$, taken along contours, laying inside the disc D, connecting the points 1 and z and not passing through zero, converge. For $\varphi \in \Phi$ introduce the functions

$$b_\varphi^{(l)}(z,\varsigma) = \exp\left\{ \int_1^{b_0^{(l)}} \frac{\varphi(t)}{t} dt \right\} \quad (l=1,2), \tag{8}$$

where $z,\varsigma \in D$ when $l=1$, and $z,\varsigma \in G$ when $l=2$, connecting the points 1 and $b_0^{(l)} = b_0^{(l)}(z,\varsigma)$ $(l=1,2,3)$ and not passing through zero at $z \neq \varsigma$. Obviously, in (8) instead of functions $b_0^{(l)}(z,\varsigma)$ one may take other functions, analytic in D or G.

2. The function

$$E(z,q) = \begin{cases} 1-z, & q=0, \\ (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^q}{q}}, & q \geq 1. \end{cases}$$

is called the Weierstrass primary multiplier. Let $\alpha = \{\alpha_k\}_1^\infty$ be a sequence of complex numbers, and the series $\sum_{k=1}^{\infty} \frac{\alpha_k}{k} z^k$ converges at $|z| < \infty$. Consider the entire function

$$E_\alpha(z)=(1-z)\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{k}z^k\right\}.$$

If $\alpha_k=1$ for $k=1,2,...,q$, and $\alpha_k=0$ for $k=q+1,q+2,...$, then $E_\alpha(z)\equiv E(z,q)$.

Theorem 1. Let $k=q\ge 1$ be the least integer, at which α_k is not equal to 1 and the sequence α is bounded: $|\alpha_k|\le M, k=1,2,...$ Then

1) at $|z|<\frac{1}{2}$ the inequality $|\log E_\alpha(z)|\le 2M|z|^{q+1}$ holds;

2) if for the sequence $Z=\{z_n\}_1^\infty$ of complex numbers the series $\sum_{n=1}^{\infty}|z_n|^{-q-1}$ converges, then the infinite product

$$\prod_{n=1}^{\infty}E_\alpha\left(\frac{z}{z_n}\right) \tag{9}$$

in each bounded part of the plane, the zeros of which coincide with Z, converges absolutely and uniformly to an entire function.

Proof. We assume that $|z|<1-\frac{1}{2(q+1)}$. Then

$$|\log E_\alpha(z)|=\left|\sum_{k=q+1}^{\infty}\frac{\alpha_k}{k}z^k\right|\le\frac{|z|^{q+1}}{q+1}\left(|\alpha_{q+1}|+|\alpha_{q+2}|+...\right)\le\frac{M}{q+1}\frac{|z|^{q+1}}{1-|z|}\le 2M|z|^{q+1}.$$

Since $1-\frac{1}{2(q+1)}\ge\frac{1}{2}$, the statement 1) of Theorem 1 is proved.

For the proof of the statement 2) note that at $|z|<\frac{1}{2}r\le\frac{1}{2}|z_n|$ the following inequalities are valid: $\left|\log E_\alpha\left(\frac{z}{z_n}\right)\right|<2M\frac{|z|^{q+1}}{|z_n|^{q+1}}<\frac{Mr^{q+1}}{2^q|z_n|^{q+1}}$. Therefore, a series $\sum_{n=n_0}^{\infty}\left|\log E_\alpha\left(\frac{z}{z_n}\right)\right|$ converges uniformly in $|z|<\frac{1}{2}r$, if n_0 is large enough, and, therefore, the product (9) converges absolutely and uniformly.

3. Let $\varphi\in\Phi$ and $\varphi(t)=1+\sum_{n=1}^{\infty}\beta_n t^n$, $|t|<r$ $(r>1)$.

Theorem 2. Functions $b_\varphi^{(l)}=b_\varphi^{(l)}(z,\varsigma)$, $l=1,2$, can be represented as $b_\varphi^{(l)}(z,\varsigma)=E_\alpha(1-b_0^{(l)}(z,\varsigma))$, where $\alpha_k=(-1)^k\sum_{n=k}^{\infty}\beta_n C_{n-1}^{k-1}$, $C_n^m=\frac{n(n-1)\cdots(n-m+1)}{m!}$.

Proof. For $|t|<r-1$ we have

$$\frac{\varphi(t)-1}{t}=\sum_{n=1}^{\infty}\beta_n t^{n-1}=\sum_{n=1}^{\infty}\beta_n(1-(1-t))^{n-1}=\sum_{n=1}^{\infty}\beta_n\sum_{k=0}^{n-1}C_{n-1}^k(-1)^k(1-t)^k=$$

$$=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{n=k+1}^{\infty}\beta_n C_{n-1}^k\right)(1-t)^k=\sum_{k=0}^{\infty}\alpha_{k+1}(1-t)^k .$$

Applying term by term integration over the contours, laying inside the disc D and connecting the points 1 and z ($|z|<r-1$), we obtain

$$\int_1^z\frac{\varphi(t)-1}{t}dt=\sum_{k=0}^{\infty}\frac{\alpha_{k+1}}{k+1}(1-z)^{k+1}=\sum_{k=1}^{\infty}\frac{\alpha_k}{k}(1-z)^k .$$

Thus, if $|z|<r-1$, then

$$E_\alpha(1-z)=z\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{k}(1-z)^k\right\}=z\exp\left\{\int_1^z\frac{\varphi(t)-1}{t}dt\right\}=\exp\left\{\int_1^z\frac{\varphi(t)}{t}dt\right\}. \quad (10)$$

Now, the statement of Theorem 2 follows from (8), (10) and the uniqueness theorem of analytical functions.

Example 1. Let p be natural number and $\varphi(t)=(1-t)^p$. Then $\varphi(t)=1+\sum_{n=1}^{p}(-1)^n C_p^n t^n$, $\beta_n=(-1)^n C_p^n$ when $n\le p$, and $\beta_n=0$ when $n>p$, $\alpha_{k+1}=(-1)^k\sum_{n=k+1}^{p}\beta_n C_{n-1}^k$.

On the other hand, $\dfrac{\varphi(t)-1}{t}=\dfrac{(1-t)^p-1}{t}=-((1-t)^{p-1}+(1-t)^{p-2}+\ldots+1)$, i.e. $\alpha_{k+1}=-1$ when $k=0,1,\ldots,p-1$, and $\alpha_{k+1}=0$ when $k=p,p+1,\ldots$ Thus, we get $b_\varphi^{(l)}(z,\varsigma)=E(1-b_0^{(l)}(z,\varsigma),p-1)$, $l=1,2$. This representation is proved by M.M. Djrbashyan [3] (see also [7]).

4. Let $\varphi\in\Phi$ and $\varphi(t)=1+\sum_{n=1}^{\infty}\beta_n t^n$, $|t|<1$. From (8) it follows that for functions $b_\varphi^{(l)}=b_\varphi^{(l)}(z,\varsigma)$, $l=1,2$, representations are true

$$b_\varphi^{(l)}(z,\varsigma)=b_0^{(l)}(z,\varsigma)\exp\left\{\beta_0+\sum_{n=1}^{\infty}\frac{\beta_n}{n}\left(b_0^{(l)}(z,\varsigma)^n\right)\right\}, \quad (11)$$

where the limit $\beta_0=-\lim\limits_{\substack{t\to1\\|t|<1}}\sum_{n=1}^{\infty}\frac{\beta_n}{n}t^n$ exists due to the definition of class Φ.

Example 2. Let $\alpha>-1$ be any real number and $\varphi(t)=(1-t)^\alpha$. Then $(1-t)^\alpha=1+\sum_{n=1}^{\infty}\beta_n^{(1)}(\alpha)t^n$, where $\beta_n^{(1)}(\alpha)=(-1)^n\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ $(n\ge1)$.

Note that $\beta_n^{(1)}(\alpha)\ge0$ for $-1<\alpha\le0$, and $\beta_n^{(1)}(\alpha)\le0$ for $0<\alpha\le1$.

Example 3. Let $\alpha>-1$ be any real number and $\varphi(t)=\left(\dfrac{1-t}{1+t}\right)^\alpha$. Then $\left(\dfrac{1-t}{1+t}\right)^\alpha=1+\sum_{n=1}^{\infty}\beta_n^{(2)}(\alpha)t^n$, where $\beta_1^{(2)}(\alpha)=-2\alpha$,

$$\beta_n^{(2)}(\alpha)=\frac{(-1)^n}{n}\Bigg(\sum_{k=1}^{n-1}C_n^k\alpha(\alpha-1)\cdots(\alpha-k+1)\alpha(\alpha+1)\cdots(\alpha+n-k-1)+$$
$$+\alpha(\alpha-1)\cdots(\alpha-n+1)+\alpha(\alpha+1)\cdots(\alpha+n-1)\Bigg),\ n\geq 2.$$

There holds also the recurrent formula

$$\beta_n^{(2)}(\alpha)=((1-)^n-1)\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}-\sum_{k=1}^{n-1}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\beta_{n-k}^{(2)}\quad(n\geq 2).$$

Note that $\beta_n^{(2)}(\alpha)\geq 0$ for $-1<\alpha\leq 0$, and $\operatorname{sgn}\beta_n^{(2)}(\alpha)=(-1)^n$ for $0<\alpha\leq 1$, $n\geq 1$.

From the following obvious inequalities

$$\sum_{n=1}^{\infty}\frac{|\beta_n|}{n}\operatorname{Re}(b_0^{(l)}(z,\varsigma))^n\leq\sum_{n=1}^{\infty}\frac{|\beta_n|}{n}\left|b_0^{(l)}(z,\varsigma)\right|^n\leq\lim_{t\to 1-0}\sum_{n=1}^{\infty}\frac{|\beta_n|}{n}t^n$$

and formulas (7) we obtain $\left|b_\varphi^{(l)}(z,\varsigma)\right|\leq\left|b_0^{(l)}(z,\varsigma)\right|$ when $\beta_n\geq 0\ (n\geq 1)$,

$\left|b_\varphi^{(l)}(z,\varsigma)\right|\geq\left|b_0^{(l)}(z,\varsigma)\right|$ when $\beta_n\leq 0\ (n\geq 1)$.

Particularly, due to (1), (2), (3) the following inequalities hold:

$$\left|b_\alpha^{(l)}(z,\varsigma)\right|\leq\left|b_0^{(l)}(z,\varsigma)\right|\ \text{ when }\ -1<\alpha\leq 0\ (l=1,2,3),\tag{12}$$

$$\left|b_\varphi^{(l)}(z,\varsigma)\right|\geq\left|b_0^{(l)}(z,\varsigma)\right|\ \text{ when }\ 0<\alpha\leq 1\ (l=1,3).$$

The inequality (12) in case $l=2$ is established in [5] with the help of other method.

If $\left|1-b_0^{(l)}(z,\varsigma)\right|<1$, then writing the functions $b_0^{(l)}=b_\varphi^{(l)}(z,\varsigma)$ in the form $b_\varphi^{(l)}(z,\varsigma)=\exp\left\{\int_0^{\left|1-b_0^{(l)}\right|}\frac{\varphi(1+re^{i\vartheta})}{1+re^{i\vartheta}}e^{i\vartheta}dr\right\}$, where $\vartheta=\arg(b_0^{(l)}-1)$, we obtain the inequalities

$$\left|\log b_\varphi^{(l)}(z,\varsigma)\right|\leq\frac{1}{1-\left|1-b_0^{(l)}\right|}\int_0^{\left|1-b_0^{(l)}\right|}\left|\varphi(1+re^{i\vartheta})\right|dr\quad(l=1,2).\tag{13}$$

Assume that $|\varphi(t)|=O(|1-t|^\rho)$ with $\rho>0$ holds for $t\to 1$, $|t|<1$. Then, due to (13) the inequality holds

$$\left|\log b_\varphi^{(l)}(z,\varsigma)\right|\leq O(1)\frac{\left|1-b_0^{(l)}(z,\varsigma)\right|^{1+\rho}}{1-\left|1-b_0^{(l)}\right|}.\tag{14}$$

For $\varsigma\in D$, $|z|\leq R<1$ we have

$$\left|1-b_0^{(1)}(z,\varsigma)\right|=\frac{1-|\varsigma|^2}{|1-\bar{\varsigma}z|}\leq\frac{2}{1-R}(1-|\varsigma|);\tag{15}$$

for $\varsigma\in G$, $\operatorname{Im}z\leq-\rho<0$ we have

$$\left|1-b_0^{(2)}(z,\varsigma)\right|=\frac{2|\operatorname{Im}\varsigma|}{|z-\varsigma|}\leq\frac{2}{\rho}|\operatorname{Im}\varsigma|\ .\tag{16}$$

From inequalities (14), (15), (16) the convergence conditions (4), (6) of infinite products (5), (7) follow immediately.

Example 4. The case of analytic function φ with singularity in a point $z=1$ is of interest. Let $\varphi(t)=e^{\frac{t}{t-1}}=e\cdot e^{\frac{1}{t-1}}$, $t\in D$. From (13), (16) in the case $z,\varsigma\in G$, $|\operatorname{Im} z|\geq\rho>0$, $|\operatorname{Im}\varsigma|\leq\frac{\rho}{3}$ we get the following estimation:

$$\left|\log b_{\varphi}^{(2)}(z,\varsigma)\right|\leq 3e\int_{0}^{\left|1-b_0^{(2)}\right|} e^{\frac{\cos\vartheta}{r}}\,dr\leq 3\left|b_0^{(2)}(z,\varsigma)-1\right|\exp\left\{1+\frac{\operatorname{Re}(b_0^{(2)}-1)}{\left|b_0^{(2)}(z,\varsigma)-1\right|^2}\right\}\leq$$

$$\leq\frac{6\sqrt{e}}{\rho}\left|\operatorname{Im}\varsigma\right|\exp\left\{-\frac{1}{2}\frac{|\operatorname{Im} z|}{|\operatorname{Im}\varsigma|}\right\}.$$

Thus, if for the sequence $\{z_n\}_1^{\infty}\subset G$ $\left(\lim\limits_{n\to\infty}\operatorname{Im} z_n=0\right)$ for any $\rho>0$ holds

$$\sum_{n=1}^{\infty} e^{-\frac{\rho}{|\operatorname{Im} z_n|}}<+\infty, \tag{17}$$

then the infinite product $B_{\varphi}^{(2)}(z,\{z_n\})=\prod\limits_{n=1}^{\infty}\exp\left\{\int\limits_{1}^{b_0^{(2)}(z,z_n)} e^{\frac{t}{t-1}}\frac{dt}{t}\right\}$ converges absolutely and uniformly in any half-plane $\{z:\operatorname{Im} z<-\rho\}$, and represents functions analytic in half-planes G with zeros $\{z_n\}_1^{\infty}$. For example, if $|\operatorname{Im} z_n|=\frac{1}{\log^2 n}$, then the condition (17) holds, but none of conditions (6) is satisfied.

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Վայերշտրասի և Բլյաշկեի տիպի ֆունկցիաներ

Աշխատանքում ընդհանրացված են Վայերշտրասի արտադրիչները և ապացուցված է համապատասխան անվերջ արտադրյալների զուգամի-տության թեորեմը: Ապացուցված են Բլյաշկեի տիպի ֆունկցիաների ներկայացումներ Վայերշտրասի տիպի և Բլյաշկեի ֆունկցիաների միջոցով: Բերված է Բլյաշկեի տիպի ֆունկցիաների կառուցման և դրանց զուգամիտությունն ապացուցելու մեթոդ: Ստացված են որոշ առնչություններ Բլյաշկեի տիպի արտադրյալների միջև:

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Функции типа Вейерштрасса и Бляшке

В данной статье обобщены множители Вейерштрасса и доказана теорема сходимости соответствующих бесконечных произведений. Установлены представления функций типа Бляшке через функции типа Вейерштрасса и функции Бляшке. Указаны способ построения новых произведений типа Бляшке и метод доказательства их сходимости. Установлены некоторые соотношения между произведениями типа Бляшке.