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Mathematics

NEUMANN PROBLEM FOR FOURTH ORDER DEGENERATE ORDINARY DIFFERENTIAL EQUATIONS

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In the present paper the Neumann problem for the equation $Lu \equiv \left(t^{\alpha}u''\right)'' + au = f$, where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, is considered. Firstly, the weighted Sobolev space W_{α}^{2} and generalized solution for the above-mentioned equation are defined. Then, the existence and uniqueness of the generalized solution is studied, as well as the spectrum and the domain of corresponding operator are described.

***Keywords*:** Neumann problem, weighted Sobolev spaces, generalized solution, spectrum of linear operators.

The Problem Formulation. Consider the Neumann problem for the following ordinary differential equation of the fourth order

$$Lu \equiv \left(t^{\alpha}u''\right)'' + au = f, \tag{1}$$

where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, $a = const$.

Define the weighted Sobolev space W_{α}^{2} and consider behavior of functions from W_{α}^{2} in neighborhood of $t = 0$. Then we define the generalized solution to the Neumann problem for the equation (1). Under some conditions on factor a we have to prove the existence and uniqueness of the generalized solution. Moreover, we have to give a description of the spectrum of operator L and the operator domain $D(L)$. Note that the Dirichlet problem for degenerate ordinary differential equations of second and fourth orders have been considered in [1, 2] and for higher orders – in [3].

The Space W_{α}^{2}. Let $\alpha \ge 0$ and t belong to the finite interval $(0,b)$. Consider the set W_{α}^{2} of the functions $u(t)$, which have generalized derivative of

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the second order, such that the following semi-norm $\|u\|_1 = \int_0^b t^\alpha |u''(t)|^2\, dt$ is finite. First note, that for the functions $u \in W_\alpha^2$ for every $t_0 \in (0,b]$ there exist finite values $u(t_0)$ and $u'(t_0)$ (see [3]). Below we study the behavior of functions $u(t)$ and $u'(t)$ in neighborhood of $t = 0$.

Proposition 1. For $u \in W_\alpha^2$ the following inequalities hold:

$$|u(t)|^2 \le (c_1 + c_2 t^{3-\alpha})\|u\|_1^2,\ \alpha \ne 1,\ \alpha \ne 3, \tag{2}$$

$$|u'(t)|^2 \le (c_1 + c_2 t^{1-\alpha})\|u\|_1^2,\ \alpha \ne 1. \tag{3}$$

In (2) $t^{3-\alpha}$ is replaced with $|\ln t|$ for $\alpha = 3$, $t^{3-\alpha}$ – with $t^2|\ln t|$ for $\alpha = 1$. In (3) $t^{1-\alpha}$ is replaced with $|\ln t|$.

The proof is carried out analogously to those in [2] and [4]. Namely we use the representation $u(t) = u(t_0) + \int_{t_0}^t u'(\tau)d\tau$ and apply the Cauchy inequalities.

From Proposition 1 it follows that for $0 \le \alpha < 1$ (weak degeneration) the values $u(0)$ and $u'(0)$ are finite, for $1 \le \alpha < 3$ only $u(0)$ is finite, while for $\alpha \ge 3$ both $u(0)$ and $u'(0)$ may turn to infinity. From the inequality (2) for $0 \le \alpha < 4$ we get the inequality

$$\|u\|_{L_2(0,b)} \le c\|u\|_1, \tag{4}$$

i.e. we have the following inclusion

$$W_\alpha^2 \subset L_2(0,b). \tag{5}$$

Inclusion (5) is true also for $\alpha = 4$. Indeed, using Hardy's inequality (see [5]), we get

$$\int_0^b |u(t)|^2\, dt = \int_0^b \left| u(b) + \int_t^b u'(\tau)d\tau \right|^2 dt \le c\int_0^b t^2\, |u'(t)|^2\, dt,$$

$$\int_a^b t^2\, |u'(t)|^2\, dt = \int_0^b t^2 \left| u'(b) + \int_t^b u''(\tau)d\tau \right|^2 dt \le (c_1 + c_2)\int_0^b t^4\, |u''(t)|^2\, dt.$$

Inclusion (5) for $\alpha > 4$ doesn't hold. Indeed, the function $u(t) = t^{-\frac{1}{2}}$ belongs to W_α^2 for $\alpha > 4$, but $u \notin L_2(0,b)$. Therefore, to remain in frames of $L_2(0,b)$, we further assume that $0 \le \alpha \le 4$. Now we can define the following norm in W_α^2:

$$\|u\|_{W_\alpha^2}^2 = \int_0^b \left(t^\alpha |u''(t)|^2 + |u(t)|^2 \right) dt. \tag{6}$$

The space W_α^2 is a Hilbert space with scalar product $(u,v)_\alpha = (t^\alpha u'', v'') + (u,v)$, where $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,b)$. Obviously, for $0 \le \alpha \le 4$ we have the following inequality

$$\|u\|_{L_2(0,b)} \le c\|u\|_{W_\alpha^2} . \tag{7}$$

Proposition 2. The inclusion (4) for $0 \le \alpha < 4$ is compact.

Indeed, using inequality (3) we get

$$\|u(t+h)-u(t)\|_{L_2(0,b)}^2 = \int_0^b |u(t+h)-u(t)|^2\, dt = \int_0^b \left| \int_t^{t+h} u'(\tau) d\tau \right|^2 dt \le$$

$$\le \int_0^b \left| \int_t^{t+h} \left(\sqrt{c_1} + c_3 t^{\frac{1-\alpha}{2}} \right) d\tau \right|^2 dt \cdot \|u\|_1^2 = \int_0^b \left(\sqrt{c_1} h + c_4 \left((t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}} \right) \right)^2 dt \cdot \|u\|_1^2 \le$$

$$\le \left(c_5 |h| + 2c_4 \int_0^b \left| (t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}} \right|^2 dt \right) \cdot \|u\|_1^2 \le c|h|^\beta \|u\|_1^2 ,$$

i.e. we have the following inequality: $\|u(t+h)-u(t)\|_{L_2(0,b)} \le c\|u\|_{W_\alpha^2}$.

The result now follows from the pre-compactness criterion in $L_2(0,b)$. Note also that for $\alpha = 4$ the continuous inclusion (5) is not compact (see [1]).

The Neumann Problem.

Definition 1. The function $u \in W_\alpha^2$ is called the generalized solution to Neumann problem for equation (1), if for every $v \in W_\alpha^2$ we have the equality

$$(t^\alpha u'', v'') + a(u,v) = (f,v). \tag{8}$$

Note that, if the generalized solution $u \in W_\alpha^2$ is classical, then for $\alpha = 0$ we get the following conditions (see [6]): $u''(0) = u'''(0) = u''(b) = u'''(b) = 0.$ Consider the particular case of the equation (1) when $a = 1$.

$$Bu \equiv \left(t^\alpha u''\right)'' + u = f. \tag{9}$$

Proposition 3. For every $f \in L_2(0,b)$ the generalized solution of Neumann problem for equation (9) exists and is unique.

Proof. Uniqueness of the generalized solution of equation (9) immediately follows from equality (8) (with $a = 1$), if we put $f = 0$ and $v = u.$ To prove the existence define the functional $l_f(v) = (f,v),\ f \in L_2(0,b)$, over the space W_α^2. Using the inequality (7) we get

$$|l_f(v)|^2 = \left| \int_0^b f(t)\overline{v(t)} dt \right|^2 \le \|f\|_{L_2(0,b)}^2 \|v\|_{L_2(0,b)}^2 \le c\|f\|_{L_2(0,b)}^2 \|v\|_{W_\alpha^2} ,$$

i.e. $l_f(v)$ is a linear continuous functional over the space W_α^2. Using Riesz lemma on representation we get $l_f(v) = (u_0, v)_\alpha,\ u_0 \in W_\alpha^2$. Therefore, the function u_0 is the generalized solution to equation (9) (see [1]).

Define the operator $B : L_2(0,b) \to L_2(0,b)$ corresponding to Definition 1.

Definition 2. We say that the function $u \in W_\alpha^2$ belongs to the domain $D(B)$ of operator B, if there exists $f \in L_2(0,b)$ such that the equality (7) is valid. In this case we write $Bu = f$.

Theorem 1. The operator $B: L_2(0,b) \to L_2(0,b)$ is positive and self-adjoint. The bounded operator $B^{-1}: L_2(0,b) \to L_2(0,b)$ for $0 \le \alpha < 4$ is compact.

Proof. The symmetry and positivity of operator B is a direct consequence of Definition 2. The coincidence of $D(B)$ and $D(B^*)$ (B^* is the adjoint to operator B) follows from the existence of a generalized solution of (9) for every $f \in L_2(0,b)$ (see Proposition 3). Note that Definition 2 implies the inequality $\|u\|_{W_\alpha^2} \le c\|Bu\|_{L_2(0,b)}$. Compactness of the operator B^{-1} for $0 \le \alpha < 4$ now follows from the Proposition 2.

Corollary. For $0 \le \alpha < 4$ the operator B has discrete spectrum, and its eigenfunction system is complete in $f \in L_2(0,b)$ (see [7]).

Note that now we can rewrite the equation (1) in the form $Bu = (1-a)u + f$, i.e. we can refer the number $1-a$ as a spectral parameter.

Theorem 2. The domain of operator L consists of functions $u(t,)$ for which $u(0)$ is finite when $0 \le \alpha < 7/2$ and $u'(0)$ is finite for $0 \le \alpha < 2$. The values $u(0)$ and $u'(0)$ can not be specified arbitrarily, but are determined by the right-hand side of (1).

Proof. Since $D(L) = D(L - aI)$, it is sufficient to study the properties of Neumann problem for equation

$$\left(t^\alpha u''\right)'' = f. \tag{10}$$

Let $1 \le \alpha < 2$. The derivative of the general solution to equation (10) has the following form:

$$u'(t) = c_1 + c_2 t^{2-\alpha} + \int_0^t \tau^{-\alpha} \int_0^\tau (\tau - \eta) f(\eta) d\eta d\tau.$$

We have $\left| \int_0^t \tau^{-\alpha} \int_0^\tau (\tau-\eta) f(\eta) d\eta d\tau \right| \le c t^{\frac{5}{2}-\alpha} \|f\|_{L_2(0,b)}$.

For $\alpha \ge 2$ the value $u'(0)$, generally speaking, can be infinite. Now assume $3 \le \alpha < 7/2$. Then we can write the solution in the following form:

$$u(t) = c_3 + c_4 t - \int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta - \xi) f(\xi) d\xi d\eta d\tau .$$

Then we have

$$\left| \int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta-\xi) f(\xi) d\xi d\eta d\tau \right| \le c \left| t^{\frac{7}{2}-\alpha} - \left(\frac{7}{2} - \alpha\right) b^{\frac{5}{2}-\alpha} t \right| \|f\|_{L_2(0,b)},$$

which completes the proof.

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REFERENCES

1. **Dezin A.A.** Mat. Sbornik, 1982, v. 43, № 3, p. 287–298 (in Russian).
2. **Tepoyan L.** Differential'nye Urawneniya, 1987, v. 23, № 8, p. 1366–1376 (in Russian); English transl. in Amer. Math. Society, 1988, v. 23, № 8, p. 930–939.
3. **Tepoyan L.P.** Izv. NAN Armenii. Matematika, 1999, v. 34, № 5, p. 48–56 (in Russian).
4. **Kudryavtzev L.D.** Trudy Mat. Inst. AN SSSR, 1984, v. 170, p. 161–190 (in Russian).
5. **Hardy G.H., Littlewood J.E., Polya G.** Inequalities. Cambridge: Cambridge University Press, 1964.
6. **Showalter R.E.** Hilbert Space Methods for Partial Differential Equations, 1994, www.emis.de/journals/EJDE/Monographs/01/abstr.html
7. **Dezin A.A.** Partial Differential Equations (An Introduction to a General Theory of Linear Boundary Value Problems). Springer, 1987.

Լ. Պ. Տեփոյան, Դարյուշ Քալվանդ

Նեյմանի խնդիրը չորրորդ կարգի սովորական վերասերվող դիֆերենցիալ հավասարումների համար

Աշխատանքում դիտարկվում է Նեյմանի խնդիրը հետևյալ հավասարման համար. $Lu \equiv (t^{\alpha} u'')'' + au = f$, որտեղ $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$: Նախ սահմանվում են Սոբոլևի կշռային տարածությունը և այդ հավասարման ընդհանրացված լուծումը: Այնուհետև հետազոտվում են ընդհանրացված լուծման գոյության և միակության հարցերը, ինչպես նաև տրվում են համապատասխան օպերատորի սպեկտրի և որոշման տիրույթի նկարագրերը:

Л. П. Тепоян, Дарюш Калванд

Задача Неймана для обыкновенного вырождающегося дифференциального уравнения четвертого порядка

В работе рассматривается задача Неймана для уравнения $Lu \equiv (t^{\alpha} u'')'' + au = f$, где $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$. Сначала мы определяем весовое пространство Соболева W_{α}^{2} и обобщенное решение для этого уравнения. Затем изучается вопрос существования и единственности обобщенного решения, а также даются описания спектра и области определения для соответствующего оператора.