PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2010, № 1, p. 58–60

COMMUNICATIONS

Mathematics

EFFECTIVE ESTIMATES FOR MODEL GENERATED BY DISTRIBUTION OF MODERATE GROWTH

H. G. HOVHANNISYAN*

Chair of Probability Theory and Mathematical Statistics, YSU

In this note two-parametric model, generated by distribution of moderate growth, is considered. The parametric function of unknown parameters, for which there is unbiased, effective estimate, is obtained. The estimate itself is built too.

***Keywords*:** distribution of moderate growth, parameters' estimation, effective estimate.

1^0. Introduction. In order to obtain an *effective* estimates for unknown parameter $\alpha=(\theta,c)$ with parametric set $A=\{\alpha=(\theta,c):0<\theta\le 1, c>0\}$ in two-parametric distribution of moderate growth (1)–(3) of paper [1], replacing $\ln\left(1+\dfrac{c-1}{\psi_k}\right)$ by $(c-1)/\psi_k$ we get the following distribution $\{p_\alpha(x)\}_0^\infty$:

$$\begin{cases} p_\alpha(x)=(g(\alpha))^{-1}\dfrac{\theta^x}{\psi_x}\exp\left\{(c-1)\sum\limits_{m=0}^{x-1}\dfrac{1}{\psi_m}\right\}, \quad x=1,2,\dots, \\ p_\alpha(0)=(g(\alpha))^{-1}=\left(1+\sum\limits_{x\ge1}\dfrac{\theta^x}{\psi_x}\exp\left\{(c-1)\sum\limits_{m=0}^{x-1}\dfrac{1}{\psi_m}\right\}\right)^{-1}. \end{cases} \tag{1.1}$$

The moderate growth is defined by conditions:

$$\psi_0=1,\ \{\psi_x\}_0^\infty \text{ increases},\quad \lim_{x\to+\infty}(\psi_k/\psi_{k-1})=1,\quad S_\psi=\sum_{x\ge1}\left(1/\psi_k\right)<+\infty. \tag{1.2}$$

We build the model (1.1)–(1.2), because the distribution $\{p_\alpha(x)\}_0^\infty$ of random variable (RV) $\xi\ge0$ belongs to two-parametric *exponential* class, i.e. the representation holds (see [2])

$$p_\alpha(x)=h(x)\exp\left\{\sum_{i=1}^{2}A_i(\alpha)\cdot T_i(x)+B(\alpha)\right\}, \tag{1.3}$$

where all functions are finite and measurable with respect to corresponding variables. Namely, in our case

$$h(x)=(\psi_x)^{-1}\exp\{-S_\psi(x)\},\quad T_1(x)=x,\quad T_2(x)=S_\psi(x),\quad A_1(\alpha)=\ln\theta,\quad A_2(\alpha)=c,$$

* E-mail: Hovhannisyan@list.ru

$B(\alpha)=-\ln g(\alpha)$. Here we denote $S_\psi(x)=\sum_{m=0}^{x-1}(1/\psi_m)$.

The belongings to exponential class allows, based on known facts of Mathematical Statistics, to find the *unique* parametric function $\tau(\alpha)=(\tau_1(\alpha),\tau_2(\alpha))$, which has *unbiased*, *effective* estimate $\tau^*=(\tau_1^*,\tau_2^*)$ for distribution (1.1)–(1.2).

Let $X^n=(X_1,X_2,...,X_n)$ be a sample of size $n>1$ of RV ξ, $\overline{X}^n=\frac{1}{n}\sum_{i=1}^n X_i$, $\overline{S}_\psi(X^n)=\frac{1}{n}\sum_{i=1}^n S_\psi(X_i)$. Note that the statistics $\left(\sum_{i=1}^n X_i,\sum_{i=1}^n S_\psi(X_i)\right)$ is *complete*, *sufficient* statistics for two-dimensional parameter $A(\alpha)=(\ln\theta,c)$. It follows from the form of likelihood function $p_\alpha(X_1)\cdots p_\alpha(X_n)$ of distribution (1.1)–(1.2) and from the Lehmann Theorem ([2], p.146). Then, due to Lehmann–Sheffe Theorem 2 (see, [3], p. 74), the introduced statistics is *unbiased*, *optimal* estimate for parametric function $\tau(\alpha)=(E_\alpha\xi,E_\alpha S_\psi(\xi))$, where E denotes the sign of mathematical statistics. The effective estimate is also optimal. But the reverse statement, generally speaking, is not true. Anyway, in our case the optimal estimate is also effective, which establishes the following

Theorem. For distribution (1.1)–(1.2) there is the unique parametric function, which has unbiased, effective estimate τ^*. Moreover,

$$\tau(\alpha)=(E_\alpha\xi,E_\alpha S_\psi(\xi)),\ \tau^*=(\overline{X}^n,\overline{S}_\psi(X^n)). \tag{1.4}$$

2⁰. Method of Analysis. Introduce the following *Conditions* (*R*):

a) $p_\alpha(x)$ is continuously differentiable by $\alpha\in A$ for any $x=0,1,2...$

b) The set $\{x\in R: p_\alpha(x)>0\}$ doesn't depend on α.

c) Denote $I_{i,j}(\alpha)=E_\alpha\left(\frac{\partial\ln p_\alpha(X_1)}{\partial\alpha_i}\cdot\frac{\partial\ln p_\alpha(X_1)}{\partial\alpha_j}\right), i,j=1,2,\ \alpha_1=\theta,\ \alpha_2=c.$

Then the matrix $I(\alpha)=(I_{i,j}(\alpha))$ is continuous by α and $\det I(\alpha)\neq 0$. Denote $A(\alpha)=(A_1(\alpha),A_2(\alpha)),\ A_{i,j}(\alpha)=\partial A_i(\alpha)/\partial\alpha_j,\ i,j=1,2,\ B'(\alpha)=\left(\frac{\partial B(\alpha)}{\partial\alpha_1},\frac{\partial B(\alpha)}{\partial\alpha_2}\right)$. Then

$$\begin{cases} A'(\alpha)=(A_{i,j}(\alpha))=\begin{pmatrix}1/\theta & 0\\ 0 & 1\end{pmatrix},\quad (A'(\alpha))^{-1}=\begin{pmatrix}\theta & 0\\ 0 & 1\end{pmatrix} \text{ is reverse to } A'(\alpha) \text{ matrix},\\ \frac{\partial B(\alpha)}{\partial\alpha_1}=-\frac{1}{g(\alpha)}\cdot\frac{\partial g(\alpha)}{\partial\theta}=-\frac{1}{\theta}E_\alpha(\xi),\quad \frac{\partial B(\alpha)}{\partial\alpha_2}=-\frac{1}{g(\alpha)}\cdot\frac{\partial g(\alpha)}{\partial c}=-E_\alpha S_\psi(\xi).\end{cases} \tag{2.1}$$

The proof of Theorem is based on following known fact for exponential class (see, for instance, [2], p. 158), satisfying *Condition* (*R*):

$$\begin{cases} \text{The unique parametric function } \tau(\alpha), \text{ which has unbiased,}\\ \text{effective estimate, and the estimate } \tau^* \text{ itself, are defined from conditions}\\ \tau(\alpha)=-B'(\alpha)\cdot(A'(\alpha))^{-1},\ \tau^*=(n^{-1}S_1,n^{-1}S_2), \text{ where } S_j=\sum_{i=1}^n T_j(X_i),\ j=1,2.\end{cases} \tag{2.2}$$

Due to definition, τ^* is *effective* estimate for $\tau(\alpha)$ if the equality holds

$$D_\alpha \tau^* = \frac{1}{n} D(\alpha) \cdot I^{-1}(\alpha) \cdot D(\alpha)^T. \quad (2.3)$$

Here $D_\alpha \tau^* = E_\alpha (\tau^* - \tau(\theta))^T \cdot (\tau^* - \tau(\theta))$, $\tau_{ij}(\alpha) = \dfrac{\partial \tau_i(\alpha)}{\partial \alpha_j}$, $i, j = 1,2$, $D(\alpha) = (\tau_{ij}(\alpha))$, T denotes the transponation sign, $I^{-1}(\alpha)$ is the matrix reverse to $I(\alpha)$.

Thus, the proof of Theorem consists in: verification of *Conditions* (*R*): evaluation of (2.2) and transformation of (2.2) to the form (1.4); verification of equality (2.3).

3^0. Proof of Theorem. Due to (2.1) and (2.2), easily verified that (2.2) and (1.4) coincide. Next, according to [2] (p. 158)

$$D_\alpha \tau^* = \frac{1}{n} D(\alpha)(A'(\alpha))^{-1}. \quad (3.1)$$

Since $\tau_{11}(\alpha) = \theta^{-1} D_\alpha \xi$, $\tau_{12}(\alpha) = \mathrm{cov}_\alpha(\xi, S_\psi(\xi))$, $\tau_{21}(\alpha) = \theta^{-1} \mathrm{cov}_\alpha(\xi, S_\psi(\xi))$, $\tau_{22}(\alpha) = D_\alpha S_\psi(\xi)$, where D is the sign of variation, cov – the sign of covariation, therefore, (3.1) takes the form

$$D_\alpha \tau^* = \frac{1}{n} \begin{pmatrix} D_\alpha \xi & \mathrm{cov}_\alpha(\xi, S_\psi(\xi)) \\ \theta^{-1} \mathrm{cov}_\alpha(\xi, S_\psi(\xi)) & D_\alpha S_\psi(\xi) \end{pmatrix}. \quad (3.2)$$

On the other hand (see [2], p. 159),

$$I(\alpha) = D(\alpha)^T A'(\alpha) = \frac{1}{\theta} \begin{pmatrix} \frac{1}{\theta} D_\alpha \xi & \mathrm{cov}_\alpha(\xi, S_\psi(\xi)) \\ \mathrm{cov}_\alpha(\xi, S_\psi(\xi)) & \theta \cdot D_\alpha S_\psi(\xi) \end{pmatrix}. \quad (3.3)$$

Due to (3.2), (3.3), the estimate (1.4) is effective. It remains to verify the fulfillment of *Condition* (*R*). Obviously, (a) and (b) take place. From Couchy–Shwartz Inequality we have

$$\det I(\alpha) = \frac{1}{\theta^2} \left(D_\alpha \xi \cdot D_\alpha S_\psi(\xi) - \mathrm{cov}_\alpha(\xi, S_\psi(\xi)) \right) > 0.$$

Thus, the condition (c) is fulfilled too.

The theorem is proved.

Received 22.12.2009

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Հ. Գ. Հովհաննիսյան

Արդյունավետ գնահատականներ չափավոր աճի բաշխումով
ծնված մոդելի համար

Աշխատանքում դիտարկված է մի երկպարամետրական մոդել, որը ծնված է չափավոր աճի բաշխումով: Ստացված է անհայտ պարամետրերի պարամետրական ֆունկցիա, որի համար գոյություն ունի անշեղելի, արդյունավետ գնահատական: Կառուցված է նշված գնահատականը:

А. Г. Оганесян.

Эффективные оценки для модели, порожденной
распределением умеренного роста

В настоящей заметке рассмотрена двухпараметрическая модель, порожденная распределением умеренного роста. Получена параметрическая функция неизвестных параметров, для которой существует несмещенная эффективная оценка. Найдена также сама оценка