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Mathematics

ON A GENERALIZATION OF TAYLOR–MACLOURIN FORMULA FOR CLASSES OF DZRBASHYAN FUNCTIONS $C_{\alpha}^{*(\infty)}$

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In the paper for any $\rho \geq 1$ and an arbitrary increasing sequence of positive numbers $\{\lambda_j\}_0^{\infty}$, the systems of operators and functions are introduced: $\left\{L_{\infty}^{n/\rho}\right\}_0^{\infty}$, $\{\varphi_n(x)\}_0^{\infty}$, $x \in [0,+\infty)$, $L_{\infty}^{0/\rho} f \equiv f$, $L_{\infty}^{n/\rho} f \equiv \prod_{j=0}^{n-1}(D_{\infty}^{1/\rho} + \lambda_j) f$, $n \geq 1$, where $D_{\infty}^{-\alpha} f \equiv \dfrac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt$, $D_{\infty}^{n/\rho} f \equiv D_{\infty}^{1/\rho} D_{\infty}^{(n-1)/\rho} f (1-\alpha = \dfrac{1}{\rho})$; $\varphi_0(x) = e^{-\lambda_0^{\rho} x}$, $\varphi_n(x) = \sum_{k=0}^{n} C_k^{(n)} e^{-\lambda_k^{\rho} x}$, $C_k^{(n)} = \left(\prod_{j=0,\,(j\neq k)}^{n} (\lambda_j - \lambda_k) \right)^{-1}$.

Some properties of these systems are investigated, as well as specific differential equations of fractional order are solved. Finally, for some classes of functions Taylor–McLaurens type formulas are obtained.

***Keywords*:** Weil operators, Taylor–McLaurens type formulas.

1. Introduction. Let $f(x)$ be an arbitrary function in the class $L(0,l)$, $0 < l < +\infty$. The function $D_{\infty}^{-\alpha} f(x) \equiv \dfrac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt$, is called the Weil integral of order α, $0 < \alpha < +\infty$, of function $f(x)$, and for $\alpha \in (0,1)$ the function $D_{\infty}^{\alpha} f(x) \equiv \dfrac{d}{dx} D_{\infty}^{-(1-\alpha)} f(x)$ is called the α-order Weil derivative of $f(x)$. It is known, that if $x^{\alpha} f(x) \in L(0,+\infty)$, then integral $D_{\infty}^{-\alpha} f(x)$ exists almost everywhere on $(0,+\infty)$ and belongs to $L(0,+\infty)$. If $f(x) \in L(0,+\infty)$ and $x^{\alpha} f(x) \in L(0,+\infty)$ for any $\alpha > 0$, then in all Lebeg points of $f(x)$ $\lim_{\alpha \to +0} D_{\infty}^{-\alpha} f(x) = f(x)$ and therefore, $\left[D_{\infty}^{-\alpha} f(x)\right]_{\alpha=0} = f(x)$ and $D_{\infty}^{1} f(x) = f'(x)$ (these properties of Weil operators are given in [1]). The operators

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$$D_\infty^0 f(x) \equiv f(x),\ \ D_\infty^{1/\rho} f(x) \equiv \frac{d}{dx} D_\infty^{-\alpha} f(x),\ \ D_\infty^{n/\rho} f(x) \equiv D_\infty^{1/\rho} D_\infty^{(n-1)/\rho} f(x),\ n=2,3,\ldots,$$

are called Weil operators of successive differentiation of order n/ρ for function $f(x)$. In particular, $D_\infty^{n/\rho} f(x) \equiv f^{(n)}(x)$ in case of $\alpha = 0,\ \rho = 1$.

In [2] the following classes of functions are introduced:

- $C_\alpha^{(\infty)} (0 \le \alpha < 1)$ is the class of functions $f(x)$, possessing all successive derivatives $D_\infty^n f(x) \equiv f^{(n)}(x),\ n = 0,1,\ldots$, satisfying conditions

$$\sup_{0 \le x < \infty} |(1 + x^{m\alpha}) f^{(n)}(x)| < +\infty,\quad n, m = 0,1,2,\ldots;$$

- $C_\alpha^{*(\infty)} (0 \le \alpha < 1)$ is the class of functions $f(x)$, possessing all successive Weil derivatives $D_\infty^{n/\rho} f(x)\ (n = 0,1,\ldots)$, continuous on $(0,+\infty)$ and satisfying conditions

$$\sup_{0 \le x < \infty} |(1 + x^{m\alpha}) D_\infty^{n/\rho} f(x)| < +\infty,\quad n, m = 0,1,\ldots$$

In [2] it is proved that these classes coincide.

Let $\{M_n\}_1^\infty$ be an arbitrary sequence of numbers. M.M. Dzrbashyan introduce the following subclasses of functions:

- $C_\alpha^* \{[0,\infty); M_n\}$ is the subclass of functions from $C_\alpha^{*(\infty)}$, satisfying conditions $\sup\limits_{0 \le x < \infty} | D^{n/\rho} f(x)| < A \cdot B^n \cdot M_n,\quad n = 1,2,\ldots$;

- $C_\alpha \{[0,\infty); M_n\}$ is the subclass of functions from $C_\alpha^{(\infty)}$, satisfying conditions $\sup\limits_{0 \le x < \infty} |(1 + \alpha x^2) f^{(n)}(x)| \le A \cdot B^n \cdot M_n,\ \ n = 1,2,\ldots$

$A = A(f), B = B(f)$ are the constants, depending on the function $f(x)$ from the given class.

The following problem for these classes was posed by M.M. Dzrbashyan: to find conditions on $\{M_n\}_1^\infty$, such that the equality $D_\infty^{n/\rho} f(0) = 0$, $f^{(n)}(0)=0$, $n=0,1,\ldots$, implies $f(x)\equiv 0$, $x \in (0,+\infty)$. The corresponding classes were called α-quasi-analytic. Later, in 1970 M.M. Dzrbashyan formulated another relevant problem: to build a system of such functions $\{\varphi_n(x)\}_0^\infty$, with the help of which any function of a given classes may be expanded into series.

In the present work to solve this problem systems of operators and functions are built.

2. Initial Information.

2.1. Function of Mittag–Leffler type $E_\rho(z;\mu) = \sum\limits_{n=0}^{\infty} \dfrac{z^n}{\Gamma(\mu + n\rho^{-1})} (\rho > 0)$ is an entire function of order ρ and type 1 with arbitrary value of parameter μ ([2], chap. VI, § 1).

2.2. For any $\mu > 0, \alpha > 0$ the following formula holds:

$$\frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} E_\rho(\lambda t^{1/\rho}; \mu) t^{\mu-1} dt = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho}; \mu + \alpha),$$

where λ is a complex parameter, and the integration is taken along the intercept, connecting points 0 and z ([2] chap. III, (1.16)).

2.3. For any $\alpha > 0$, $\beta > 0$ the following formula holds:

$$\begin{aligned}&\int_0^l x^{\alpha-1} E_\rho(\lambda x^{1/\rho};\alpha)(e-x)^{\beta-1} E_\rho(\lambda^*(e-x)^{1/\rho};\beta)dx = \\ &= \frac{\lambda E_\rho(e^{1/\rho}\lambda;\alpha+\beta) - \lambda^* E_\rho(e^{1/\rho}\lambda^*;\alpha+\beta)}{\lambda-\lambda^*} l^{\alpha+\beta-1},\end{aligned} \tag{2.1}$$

where λ и λ^* $(\lambda \neq \lambda^*)$ are any complex parameters ([2], chap. III, (1.21), [3]).

From (2.1) using famous formula $E_\rho\left(z;\dfrac{2}{\rho}\right) = \dfrac{1}{z}\left\{E_\rho\left(z;\dfrac{1}{\rho}\right) - \dfrac{1}{\Gamma(\rho^{-1})}\right\}$ for particular case $\alpha = \beta = \dfrac{1}{\rho}$ we have

$$\begin{aligned}&\int_0^x E_\rho\left(\lambda t^{1/\rho};\frac{1}{\rho}\right) t^{\frac{1}{\rho}-1} E_\rho\left(\lambda^*(x-t)^{1/\rho};\frac{1}{\rho}\right)(x-t)^{\frac{1}{\rho}-1} dt = \\ &= \frac{E_\rho\left(\lambda x^{1/\rho};\frac{1}{\rho}\right) - E_\rho\left(\lambda^* x^{1/\rho};\frac{1}{\rho}\right)}{\lambda-\lambda^*} x^{\frac{1}{\rho}-1}.\end{aligned} \tag{2.2}$$

2.4. It is known ([1],(2.22)), that function $\tilde{e}_\rho(x;\lambda) \equiv e^{-\lambda^\rho x}$ $\left(|\arg\lambda| \le \dfrac{\pi}{2\rho}, \dfrac{1}{2} < \rho < +\infty\right)$ is the solution of the following Cauchy type equation:

$$D_\infty^{1/\rho} y(x) + \lambda y(x) = 0, \quad y(0) = 1. \tag{2.3}$$

From (2.3) it particularly follows that

$$D_\infty^{n/\rho}\left\{e^{-\lambda^\rho x}\right\} = (-\lambda)^n e^{-\lambda^\rho x}, \quad n \ge 1, \ x \in [0,+\infty). \tag{2.4}$$

2.5. It is known ([2], chap. III, (2.3); [1], (1.27')) that for $|\arg z| \le x_0$, $|z| \ge 0$

$$\left|E_\rho\left(z;\frac{1}{\rho}\right)\right| \le M_1(1+|z|)^{\rho-1} e^{\mathrm{Re}\, z^\rho} + \frac{M_2}{(1+|z|)^2}. \tag{2.5}$$

From (2.5), particularly, for $\lambda > 0$, $\rho \ge 1$ it follows that

$$E_\rho\left(\lambda x^{1/\rho};\frac{1}{\rho}\right) \le M_1(1+\lambda x^{1/\rho})^{\rho-1} e^{\lambda^\rho x} + \frac{M_2}{(1+\lambda x^{1/\rho})^2}. \tag{2.5'}$$

Lemma 2.1. Let $\rho \ge 1, \lambda > 0, e^{\lambda^\rho x}|f(x)| \in L(0,+\infty)$. Then the integral $I(x;\lambda) \equiv \int_x^\infty e_\rho(t-x;\lambda) f(t)dt$, where $e_\rho(x;\lambda) \equiv E_\rho\left(\lambda x^{1/\rho};\dfrac{1}{\rho}\right) x^{\frac{1}{\rho}-1}$, converges absolutely and uniformly along $x \in (0,\infty)$.

Proof. Due to (2.5'),

$$\left|E_\rho\left(\lambda x^{1/\rho};\frac{1}{\rho}\right)x^{\frac{1}{\rho}-1}f(x)\right|\le M_1(1+\lambda x^{1/\rho})^{\rho-1}x^{\frac{1}{\rho}-1}e^{\lambda^\rho x}\,|\,f(x)\,|+\frac{M_2x^{\frac{1}{\rho}-1}\,|\,f(x)\,|}{(1+\lambda x^{1/\rho})^2}.$$

Then, it is easy to see that

$$|\,I(x;\lambda)\,|\le\int_x^\infty\left|E_\rho\left(\lambda(t-x)^{1/\rho};1/\rho\right)\right|(t-x)^{\frac{1}{\rho}-1}\,|\,f(t)\,|\,dt\le$$
$$\le M_1\int_0^\infty(1+\lambda t^{1/\rho})^{\rho-1}t^{\frac{1}{\rho}-1}e^{\lambda^\rho t}\,|\,f(x+t)\,|dt+M_2\int_0^\infty\frac{t^{\frac{1}{\rho}-1}\,|\,f(x+t)\,|}{(1+\lambda t^{1/\rho})^2}dt, \tag{2.6}$$

but since $(1+\lambda t^{1/\rho})^{\rho-1}t^{\frac{1}{\rho}-1}\to 1,\quad \dfrac{t^{\frac{1}{\rho}-1}}{(1+\lambda t^{1/\rho})^2}\to 0$ and

$$\int_0^\infty e^{\lambda^\rho t}\,|\,f(x+t)\,|\,dt=\int_0^\infty e^{\lambda^\rho(t-x)}\,|\,f(t)\,|\,dt<\int_0^\infty e^{\lambda^\rho t}\,|\,f(t)\,|\,dt<+\infty,$$

$$\int_0^\infty|\,f(x+t)\,|\,dt=\int_x^\infty|\,f(t)\,|\,dt<\int_0^\infty|\,f(t)\,|\,dt<+\infty$$

as $t\to+\infty$, then from (2.6) follows the statement of Lemma.

Lemma 2.2. Let $\rho\ge 1,\ \lambda>0,\ f(x)\in C_\alpha^{*(\infty)},\ e^{\lambda^\rho x}\,|\,f(x)\,|\in L(0,+\infty)$. Then the function $y(x;\lambda)\equiv-\int_x^\infty e_\rho(t-x;\lambda)f(t)dt$ is the solution of the following Cauchy type problem:

$$D_\infty^{1/\rho}y+\lambda y=f(x),\quad y(\infty)=0. \tag{2.7}$$

Proof. Due to the definition of operator $D_\infty^{-\alpha}$, we have

$$D_\infty^{-\alpha}y(x;\lambda)\equiv-\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}dt\int_t^\infty e_\rho(\tau-t;\lambda)f(\tau)d\tau, \tag{2.8}$$

and since the inner integral due to Lemma 2.1 converges absolutely and uniformly along $t\in(0,+\infty)$. The change of integration order is allowed by Fubini Theorem. Then from (2.8) we get

$$D_\infty^{-\alpha}y(x;\lambda)\equiv-\frac{1}{\Gamma(\alpha)}\int_x^\infty f(\tau)dt\int_0^{\tau-x}(\tau-x-\nu)^{\alpha-1}e_\rho(\nu;\lambda)d\nu=-\int_x^\infty E_\rho(\lambda(\tau-x)^{1/\rho}1)f(\tau)d\tau. \tag{2.9}$$

Let's show that the integral of the right part of (2.9) converges absolutely and uniformly along $x\in(0,+\infty)$. It is known ([2], chap. III, (2.31)), that if $|\arg z\,|\le\beta$ and $|\,z\,|\ge 0$, then $|\,E_\rho(z;\mu)\,|\le M_1(1+|\,z\,|)^{\rho(1-\mu)}e^{\operatorname{Re}z^\rho}+\dfrac{M_2}{1+|\,z\,|}$, where $\rho>\dfrac{1}{2},\quad\beta\in\left(\dfrac{\pi}{2\rho};\min\left\{\pi;\dfrac{\pi}{\rho}\right\}\right)$ and μ is a real constant. Therefore, for $z=\lambda x^{1/\rho}$, $\mu=1$ we have $\left|E_\rho(\lambda x^{1/\rho};1)f(x)\right|\le M_1e^{\lambda^\rho x}\,|\,f(x)\,|+\dfrac{M_2\,|\,f(x)\,|}{1+\lambda x^{1/\rho}}$.

So, the integral (2.9) converges absolutely and uniformly along $x \in (0,+\infty)$.

Note that

$$\frac{d}{dx}\left\{E_\rho\left(\lambda(\tau-x)^{1/\rho};1\right)\right\} = -\lambda E_\rho\left(\lambda(\tau-x)^{1/\rho};\frac{1}{\rho}\right)(\tau-x)^{\frac{1}{\rho}-1}. \qquad (2.10)$$

Now, using formulas (2.9), (2.10) we have

$$D_\infty^{1/\rho} y(x) \equiv \frac{d}{dx} D_\infty^{-\alpha} y(x;\lambda) = \frac{d}{dx}\left\{-\int_x^\infty E_\rho(\lambda(\tau-x)^{1/\rho};1) f(\tau)d\tau\right\} =$$

$$= f(x) + \lambda \int_x^\infty E_\rho\left(\lambda(\tau-x)^{1/\rho};\frac{1}{\rho}\right)(\tau-x)^{\frac{1}{\rho}-1} f(\tau)d\tau = \qquad (2.11)$$

$$= f(x) + \lambda \int_x^\infty e_\rho(\tau-x;\lambda) f(\tau)d\tau = f(x) - \lambda y(x;\lambda).$$

(2.7) follows from (2.11).

3. Main Results. Let $\{\lambda_j\}_0^\infty$ be an arbitrary increasing sequence of positive numbers. For any $\rho \ge 1$ on proper classes of functions $f(x)$ we consider the sequence of operators:

$$L_\infty^{0/\rho} f(x) \equiv f(x), \quad L_\infty^{n/\rho} f(x) \equiv \prod_{j=0}^{n-1}(D_\infty^{1/\rho} + \lambda_j) f(x), \quad n \ge 1, \ x \in [0,+\infty). \qquad (3.1)$$

Note, that if $f(x) \in C_\alpha^{*(\infty)}$, then the functions $\left\{L_\infty^{n/\rho} f(x)\right\}_0^\infty$ exist in the interval $(0,+\infty)$, and for $n \ge 0$ $L_\infty^{n/\rho} f(x) \in C_\alpha^{*(\infty)}$, i.e.

$$\sup_{0\le x<+\infty}\left|\left(1+x^{m\alpha} D_\infty^{k/\rho} L_\infty^{n/\rho} f(x)\right)\right| < +\infty \ (n=0,1,..., \ k=0,1,..., \ m=0,1,..., \ 0 \le \alpha < 1).$$

We associate the sequence $\{\lambda_j\}_0^\infty$ with sequence of functions $\{\varphi_n(x)\}_0^\infty$ for any $\rho \ge 1$:

$$\varphi_0(x) \equiv e^{-\lambda_0^\rho x}, \quad \varphi_n(x) \equiv \sum_{k=0}^n C_k^{(n)} e^{-\lambda_k^\rho x}, \quad n \ge 1, \quad x \in [0,+\infty), \qquad (3.2)$$

where $C_k^{(n)} = \left\{\prod_{j=0,(j\ne k)}^n (\lambda_j - \lambda_k)\right\}^{-1}$, $0<\lambda_j<\lambda_{j+1}$, $j=0,1,\ldots$ Note that for any $k \ge 0$, $n \ge 0$ with the help of (2.4) we have $D_\infty^{n/\rho}\left\{e^{-\lambda_k^\rho x}\right\} = (-\lambda_k)^n e^{-\lambda_k^\rho x}$, $x \in [0,+\infty)$. From here it easily follows that

$$\sup_{0\le x<+\infty}\left|\left(1+x^{m\alpha}\right) D_\infty^{n/\rho}\left\{e^{-\lambda_k^\rho x}\right\}\right| = \sup_{0\le x<+\infty}\left|\left(1+x^{m\alpha}\right)\left(-\lambda_k\right)^n\right| < +\infty \ (0 \le \alpha < 1, m,n,k = 0,1,...).$$

Consequently, for any $k \ge 0$ holds $f_k(x) \equiv e^{-\lambda_k^\rho x} \in C_\alpha^{*(\infty)}$. Further, with the help of additivity of $C_\alpha^{*(\infty)}$ $(0 \le \alpha < 1)$, $\varphi_k(x) \in C_\alpha^{*(\infty)}$. It's easy to see, that for any $k \ge 0$ the functions $f_k(x) \in C_\alpha^*\{[0,\infty);M_n\}(M_n \ge 1)$. Indeed, $\sup_{0\le x<+\infty}\left|D_\infty^{n/\rho}\left\{e^{-\lambda_k^\rho x}\right\}\right| \le \lambda_k^n M_n$, $n=0,1,...,$ $k=0,1,...$

Due to definition (3.2), with the help of additivity of $C_\alpha^*\{[0,\infty);M_n\}$ $\varphi_k(x)\in C_\alpha^*\{[0,\infty);M_n\}$, $k=0,1,...$

Lemma 3.1. Let $\rho\ge 1$ and λ be a complex parameter, $|\arg\lambda|\le\frac{\pi}{2\rho}$. Then, for any $n\ge 1$

$$L_\infty^{n/\rho}\left\{e^{-\lambda^\rho x}\right\}=\prod_{j=0}^{n-1}(\lambda_j-\lambda)e^{-\lambda^\rho x},\ x\in[0,+\infty). \tag{3.3}$$

Proof. Using the formula $D_\infty^{1/\rho}\left\{e^{-\lambda^\rho x}\right\}=-\lambda e^{-\lambda^\rho x}$ we get

$$\left(D_\infty^{1/\rho}+\lambda_j\right)\left\{e^{-\lambda^\rho x}\right\}=\left(\left(D_\infty^{1/\rho}+\lambda\right)+\left(\lambda_j-\lambda\right)\right)\left\{e^{-\lambda^\rho x}\right\}=$$
$$\left(D_\infty^{1/\rho}+\lambda\right)\left\{e^{-\lambda^\rho x}\right\}+\left(\lambda_j-\lambda\right)e^{-\lambda^\rho x}=\left(\lambda_j-\lambda\right)e^{-\lambda^\rho x}.$$

From here, with the help of (3.1) we get (3.3).

Lemma 3.2. Let $\rho\ge 1$, $f(x)\in C_\alpha^{*(\infty)}$, $e^{\lambda^\rho x}|f(x)|\in L(0,+\infty)$. Then the function

$$y(x,\lambda_0,\lambda_1)\equiv\int_x^\infty e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^\infty e_\rho(t_1-t_0;\lambda_1)f(t_1)dt_1 \tag{3.4}$$

is the solution of the following Cauchy type problem:

$$L_\infty^{2/\rho}\,y=f(x),\ \ y(+\infty)=0, \tag{3.5}$$

where $\lambda=\max\{\lambda_0;\lambda_1\}$, $e_\rho(x,\lambda)\equiv E_\rho\left(\lambda x^{1/\rho};\frac{1}{\rho}\right)x^{\frac{1}{\rho}-1}$, $x\in(0,+\infty)$.

Proof. Note that in (3.4) the inner integral, according to Lemma 2.1, converges absolutely and uniformly along $x\in(0,+\infty)$. So, changing the integration order and using the formula (2.2), we get

$$y(x,\lambda_0,\lambda_1)=\int_x^\infty f(t_1)dt_1\int_x^{t_1}e_\rho(t_0-x;\lambda_0)e_\rho(t_1-t_0;\lambda_1)dt_0=$$
$$=\int_x^\infty f(t_1)dt_1\int_0^{t_1-x}e_\rho(\tau;\lambda_1)e_\rho(t_1-x-\tau;\lambda_0)d\tau=\int_x^\infty f(t_1)\left\{\frac{e_\rho(t_1-x;\lambda_1)-e_\rho(t_1-x;\lambda_0)}{\lambda_1-\lambda_0}\right\}dt_1,$$

i.e. $y(x,\lambda_0,\lambda_1)=\dfrac{1}{\lambda_1-\lambda_0}\left\{\int_x^\infty e_\rho(t_1-x;\lambda_1)f(t_1)dt_1-\int_x^\infty e_\rho(t_1-x;\lambda_0)f(t_1)M_1\right\}.$

Now apply the operator $L_\infty^{2/\rho}=\left(D_\infty^{1/\rho}+\lambda_0\right)\left(D_\infty^{1/\rho}+\lambda_1\right)$ to the both sides of this identity. Then, using the Lemma 2.2 we obtain (3.5):

$$L_\infty^{2/\rho}y(x,\lambda_0,\lambda_1)=\frac{1}{\lambda_1-\lambda_0}\left(D_\infty^{1/\rho}+\lambda_0\right)\left\{\left(D_\infty^{1/\rho}+\lambda_1\right)\int_x^\infty e_\rho(t_1-x;\lambda_1)f(t_1)M_1-\right.$$
$$\left.-\left(D_\infty^{1/\rho}+\lambda_1\right)\int_x^\infty e_\rho(t_1-x;\lambda_0)f(t_1)dt_1\right\}=\frac{1}{\lambda_1-\lambda_0}\left(D_\infty^{1/\rho}+\lambda_0\right)\{-f(x)+f(x)+$$

$$\left.+(\lambda_0-\lambda_1)\int_x^\infty e_\rho(t_1-x;\lambda_0)f(t_1)M_1\right\}=\left(D_\infty^{1/\rho}+\lambda_0\right)\left(-\int_x^\infty e_\rho(t_1-x;\lambda_0)f(t_1)M_1\right)=f(x).$$

Lemma 3.3. Let $\rho\ge 1$, $f(x)\in C_\alpha^{*(\infty)}$, $e^{\lambda_n^\rho x}\left|f(x)\right|\in L(0,+\infty)$. Then, for any $n\ge 1$ the function

$$y(x)\equiv(-1)^n\int_x^\infty e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^\infty e_\rho(t_1-t_0;\lambda_1)dt_1\ldots\int_{t_{n-2}}^\infty e_\rho(t_{n-1}-t_{n-2};\lambda_{n-1})f(t_{n-1})dt_{n-1}$$

is the solution of the following Cauchy type problem: $L_\infty^{n/\rho}y=f(x),\quad y(+\infty)=0$.

The Lemma is easily proved applying the method of mathematical induction.

Theorem 3.1. For any $n\ge 0$ the following relations hold:

$$1^0.\quad L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}\equiv 0,\ k\ge n+1,\ x\in\left[0,+\infty\right); \tag{3.6}$$

$$2^0.\quad L_\infty^{n/\rho}\left\{\varphi_n(x)\right\}\equiv e^{-\lambda_n^\rho x}\quad\forall n\ge 0,\ x\in\left[0,+\infty\right); \tag{3.7}$$

$$3^0.\quad L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}\big|_{x=0}=0\quad\forall\ n\ge 1,\quad 0\le k\le n-1. \tag{3.8}$$

Proof.

1^0. Note that due to definition (3.2) of functions $\varphi_n(x)$, using representation (3.1) of operators $L_\infty^{k/\rho}$ and formula (3.3), we get

$$L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}=\sum_{i=0}^n C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}. \tag{3.9}$$

But since $C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)=0$, $k\ge n+1$, for $i=0,1,...n$ (3.6) follows from (3.9).

2^0. Note that

$$L_\infty^{n/\rho}\left\{\varphi_n(x)\right\}=\sum_{i=0}^n C_i^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}=C_n^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)e^{-\lambda_n^\rho x}, \tag{3.10}$$

and since $C_n^{(n)}=\left\{\prod_{j=0,(j\ne n)}^n(\lambda_j-\lambda_n)\right\}^{-1}$ (3.7) follows from (3.10).

3^0. Assume $n\ge 1,\ 0\le k\le n-1$. Then

$$L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}=\sum_{i=k}^n C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}. \tag{3.11}$$

From (3.11) we have $L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}\big|_{x=0}=\sum_{i=k}^n C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)$. Further, since $C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)=\left\{\prod_{j=k}^n(\lambda_j-\lambda_i)\right\}^{-1}$, then

$$L_\infty^{k/\rho}\left\{\varphi_n(x)\right\}\big|_{x=0}=\sum_{i=k}^n\left\{\prod_{j=k,(j\ne i)}^n(\lambda_j-\lambda_i)\right\}^{-1}. \tag{3.12}$$

We consider the function $\varphi(\xi)=\dfrac{1}{(\xi+\lambda_k)(\xi+\lambda_{k+1})\cdots(\xi+\lambda_n)}$. $\varphi(\xi)$ is a rational function, points $\xi=-\lambda_i$, $i=k,\ k+1,...,n$, are simple poles. Using the theorems about the residues, we have $\operatorname{res}\left[\varphi(\xi)\right]_{\xi=-\lambda_i} = \left\{ \prod\limits_{j=k,(j\neq i)}^{n} (\lambda_j - \lambda_i) \right\}^{-1}$, and therefore

$$\sum_{i=k}^{n} \operatorname{res}\left[\varphi(\xi)\right]_{\xi=-\lambda_i} = 0. \tag{3.13}$$

From formula (3.12), (3.13) follows (3.8).

Theorem 3.2. The coefficients $\{a_k\}_0^n$ in the sum

$$P_n(x) = \sum_{k=0}^{n} a_k \varphi_k(x), \quad x \in [0,+\infty), \tag{3.14}$$

for any $n \ge 0$ may be determined as $a_k = L_\infty^{k/\rho} P_n(0), \ \ k = 0,1,2,...,n$.

Proof. We apply operator $L_\infty^{j/\rho}$ $(0 \le j \le n-1)$ to the function $P_n(x)$. Using (3.6) and (3.8) we find that

$$\begin{aligned} L_\infty^{j/\rho} P_n(x) &= \sum_{k=0}^{n} a_k L_\infty^{j/\rho}\{\varphi_k(x)\} = \sum_{k=j}^{n} a_k L_\infty^{j/\rho}\{\varphi_k(x)\} = a_j L_\infty^{j/\rho}\{\varphi_j(x)\} + \\ &+ \sum_{k=j+1}^{n} a_k L_\infty^{j/\rho}\{\varphi_k(x)\} = a_j e^{-\lambda_j^\rho x} + \sum_{k=j+1}^{n} a_k L_\infty^{j/\rho}\{\varphi_k(x)\}. \end{aligned} \tag{3.15}$$

Due to Theorem 3.1, $L_\infty^{j/\rho}\{\varphi_k(x)\}\big|_{x=0} = 0, \ \ k = j+1, \ j+2,...,n$, then from (3.15) we obtain $a_j = L_\infty^{j/\rho} P_n(0)$ $(0 \le j \le n-1)$.

Now we apply the operator $L_\infty^{n/\rho}$ to (3.14): $L_\infty^{n/\rho} P_n(x) = a_n L_\infty^{n/\rho}\{\varphi_n(x)\} = = a_n e^{-\lambda_n^\rho x}$, from where follows $L_\infty^{n/\rho} P_n(0) = a_n$.

Theorem 3.3. Let $f(x) \in C_\alpha^{*(\infty)}$, $e^{\lambda_{n+1}^\rho x}\left|D_\infty^{k/\rho} f(x)\right| \in L(0,+\infty), \ k = 0,1,...,n$. Then for any $n \ge 0$ $f(x) = \sum\limits_{k=0}^{n} L_\infty^{k/\rho} f(0)\varphi_k(x) + R_n(x;f), \ \ x \in [0,+\infty)$, where

$$\begin{aligned} R_n(x;f) &= (-1)^{n+1} \int_x^\infty e_\rho(t_0 - x;\lambda_0)dt_0 \int_{t_0}^\infty e_\rho(t_1 - t_0;\lambda_1)dt_1 \times \\ &\times \int_{t_{n-2}}^\infty e_\rho(t_{n-1} - t_{n-2};\lambda_{n-1})dt_{n-1} \int_{t_{n-1}}^\infty e_\rho(t_n - t_{n-1};\lambda_n) L_\infty^{\overline{n+1}/\rho} f(t_n)dt_n. \end{aligned}$$

Proof. Denote $P_n(x;f) = \sum\limits_{k=0}^{n} L_\infty^{k/\rho} f(0)\varphi_k(x), \ x \in [0,+\infty)$, and $f(x) - P_n(x;f) \equiv \equiv R_n(x;f)$. According to Theorem 3.2, we have $L_\infty^{k/\rho} P_n(x;f)\big|_{x=0} = L_\infty^{k/\rho} f(x)\big|_{x=0}$, $k = 0,1,...,n$, and $L_\infty^{\overline{n+1}/\rho} R_n(x) = L_\infty^{\overline{n+1}/\rho} f(x)$. Since $e^{\lambda_{n+1}^\rho x}\left|D_\infty^{k/\rho} f(x)\right| \in L(0,+\infty)$, $k = 0,1,...,n$, then $e^{\lambda_{n+1}^\rho}\left|L_\infty^{\overline{n+1}/\rho} f(x)\right| \in L(0,\infty)$. Further, using Lemma 3.3, we have that

$$R_n(x;f) = (-1)^{n+1} \int_x^\infty e_\rho(t_0 - x;\lambda_0)\,dt_0 \int_{t_0}^\infty e_\rho(t_1 - t_0;\lambda_1)\,dt_1 \times$$

$$\times \int_{t_{n-1}}^\infty e_\rho(t_n - t_{n-1};\lambda_n) L_\infty^{\frac{n+1}{\rho}} f(t_n)\,dt_n.$$

Thus, the Theorem is proved.

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Բ. Հ. Սահակյան, Հ. Ս. Քոչարյան

Մ.Մ. Ջրբաշյանի $C_\alpha^{*(\infty)}$ դասի ֆունկցիաների համար Թեյլոր–Մակլորենի բանաձևի մի ընդհանրացման մասին

Այս աշխատանքում ցանկացած $\rho \ge 1$ դեպքում $\{\lambda_j\}_0^\infty$-դրական թվերի կամայական աճող հաջորդականության համար մուծվում են օպերատորների և ֆունկցիաների հետևյալ համակարգերը. $\left\{L_\infty^{n/\rho}\right\}_0^\infty$, $\{\varphi_n(x)\}_0^\infty$, $x \in [0,+\infty)$:

$L_\infty^{0/\rho} f \equiv f$, $L_\infty^{n/\rho} f \equiv \prod_{j=0}^{n-1}(D_\infty^{1/\rho}+\lambda_j)f$, $n \ge 1$, որտեղ,

$$D_\infty^{-\alpha} f \equiv \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt,\ D_\infty^{n/\rho} f \equiv D_\infty^{1/\rho} D_\infty^{(n-1)/\rho} f\,(1-\alpha=\frac{1}{\rho}),\ \varphi_0(x)=e^{-\lambda_0^\rho x},$$

$$\varphi_n(x)=\sum_{k=0}^n C_k^{(n)} e^{-\lambda_k^\rho x},\ C_k^{(n)}=\left(\prod_{j=0,(j\ne k)}^n (\lambda_j-\lambda_k)\right)^{-1}:$$

Ուսումնասիրվել են այդ համակարգերի որոշ հատկություններ, դիտարկվել և լուծվել են կոտորակային կարգի որոշ դիֆերենցիալ հավասարումներ: Որոշ դասի ֆունկցիաների համար ստացվել են Թեյլոր–Մակլորենի բանաձևի տիպի բանաձևեր:

Б. А. Саакян, Г. С. Кочарян.

Об одном обобщении формулы Тейлора–Маклорена для классов функций М.М. Джрбашяна $C_\alpha^{*(\infty)}$

В работе при $\rho \ge 1$ для произвольной возрастающей последовательности положительных чисел $\{\lambda_j\}_0^\infty$ вводятся системы операторов и функций $\left\{L_\infty^{n/\rho}\right\}_0^\infty$, $\{\varphi_n(x)\}_0^\infty$, $x \in [0,+\infty)$; $L_\infty^{0/\rho} f \equiv f$, $L_\infty^{n/\rho} f \equiv \prod_{j=0}^{n-1}(D_\infty^{1/\rho}+\lambda_j)f$, $n \ge 1$, где

$$L_\infty^{1/\rho} f \equiv \frac{d}{dx} D_\infty^{-\alpha} f,\ D_\infty^{-\alpha} f \equiv \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt,\ D_\infty^{n/\rho} f \equiv D_\infty^{1/\rho} D_\infty^{(n-1)/\rho} f\,(1-\alpha=\frac{1}{\rho}),$$

$$\varphi_0(x)=e^{-\lambda_0^\rho x},\ \varphi_n(x)=\sum_{k=0}^n C_k^{(n)} e^{-\lambda_k^\rho x},\ C_k^{(n)}=\left(\prod_{j=0,(j\ne k)}^n (\lambda_j-\lambda_k)\right)^{-1}.$$

Исследуются некоторые свойства этих систем, решаются некоторые дифференциальные уравнения дробного порядка. Для функций некоторых классов получены обобщенные формулы типа Тейлора–Маклорена.