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Mathematics

PROPERTIES OF ONE LIMIT LAW IN RISK THEORY

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In the present paper some properties of one random process arising in many limit theorems of risk theory are investigated. Connection formulas with stable distributions and with one class of integral transforms are found. The asymptotical properties of this law are studied.

Keywords: limit laws, subordination, stable laws, risk theory, unimodality.

Introduction. In limit theorems of risk theory in critical situations (see [1]) the limit law with Laplace–Stieltjes transform (LST) $e^{-t\Diamond(s)}$ arises, where $\Diamond=\Delta(s)$ and $\Diamond=\nabla(s)$ are solutions of the equation $z^{\gamma}\mp z=s$ [1], $s\ge 0$, with initial conditions $\Delta(0)=0$ and $\nabla(0)=1$, respectively[2]. It is known that (see [1]) the distribution function (DF) $F_{\Diamond}(x,t)$ of random variable (RV) $\xi_{\Diamond}(t)$=$\xi_{\Diamond}(t,\gamma)$ with LST $e^{-t\Diamond(s)}$ is absolutely continuous and has density

$$f_{\Diamond}(x,t)=\operatorname{Re}\left\{\frac{1}{\pi i}\int_{-i\infty}^{0}e^{xz^{\gamma}-(t\pm x)z}(\gamma z^{\gamma-1}\mp 1)dz\right\}=$$

$$=\begin{cases}\dfrac{t}{\gamma\pi x}\displaystyle\sum_{n=0}^{\infty}\frac{(-t\mp x)^n}{n!}\Gamma\left(\frac{n+1}{\gamma}\right)x^{-\frac{n+1}{\gamma}}\sin\frac{n+1}{\gamma}\pi, & 1<\gamma<2,\\ \left(t\big/\left(2x\sqrt{\pi x}\right)\right)\exp\left\{-(t\pm x)^2/4x\right\}, & \gamma=2,\end{cases}\quad x>0,\ 0<t<\infty. \qquad (1)$$

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[1] Here and later on in signs " $\pm$ " and " $\mp$ " the upper sign is taken in the case of function ∇, and the bottom one in the case of Δ.

[2] For $\gamma=2$ we have $\Diamond(s)=\left(1\pm\sqrt{4s+1}\right)\big/2$. For some other values of parameter γ the solution $z=\Diamond(s)$ may be found as well. For example, for $\gamma=(k+1)/k$, $k=1,2,\ldots,$ and $\gamma=(2k-1)/k$, $k=2,3,\ldots,$ the equation $z^{\gamma}\mp z=s$ takes the form $y^{k+1}\mp y^k=s$ and $y^{2k-1}\mp y^k=s$. For $\gamma=1{,}5$ the function $\Diamond(s)$ may be found by means of Kardanos formula or program "Mathematika". For example, for $\gamma=1{,}5$ the function $\nabla(s)$ under condition $\nabla(0)=1$ takes the form

$$\nabla(s)=3^{-1}\left\{1+2^{1/3}(1+6s)\left(2+18s+27s^2+3\sqrt{3}\sqrt{4s^3+27s^4}\right)^{-1/3}+2^{-1/3}\left(2+18s+27s^2+3\sqrt{3}\sqrt{4s^3+27s^4}\right)^{1/3}\right\}.$$

Theorem 1. Let $t \to 0$ and $\theta \to +\infty$, so that $\theta^{1/\gamma} t \to \tau(=1)$. Then $\lim P\{\theta\xi_\Diamond(t) \le x\} = G_{1/\gamma}(x)$, where G_α, $0<\alpha<1$, is a stable DF with LST $e^{-s^{1/\gamma}}$.

The proof follows from the relation

$$\Diamond(s) = \begin{cases} s - s^\gamma(1+o(1)) = s(1+(1)), & s \to 0, \quad \Delta, \\ 1 + s(\gamma-1)^{-1}(1+o(1)), & s \to 0, \quad \nabla, \\ s^{1/\gamma} \pm \gamma^{-1} s^{2/\gamma-1}(1+o(1)) = s^{1/\gamma}(1+o(1)), & s \to \infty, \quad \Diamond, \end{cases}$$

which for the case of function Δ is presented in [4], and is easily proved for ∇.

Lemma 1[1]. $\xi_\Diamond\left(\sum_{k=1}^{n} t_k\right) \overset{d}{=} \sum_{k=1}^{n} \xi_\Diamond(t_k)$, $n \ge 2$, $t_1, t_2, \ldots, t_n > 0$, where $\{\xi_\Diamond(t_i)\}_{i=1}^{n}$ is the sequence of independent RVs, and "$\overset{d}{=}$" is the DFs equality sign.

Theorem 2[2]. The function $\Diamond(s) > 0$ has a completely monotone (CM) derivative. I. e.

$$(-1)^n(\Diamond'(s))^{(n)} \ge 0, \; s>0, \; n = 0,1,2,\ldots$$

Proof. Since the function $\phi(s) = 1/\Diamond(s)$, being a LST of some measure (see [1] and [4], p. 505), then $\Diamond'(s) = -\phi'(s)\phi^{-2}(s)$. Obviously, $\Diamond'(s) = (\gamma\Diamond^{\gamma-1}(s) \mp 1)^{-1} > 0$. Since $-\phi'(s)$ and s^{-1} are CM and $\phi^2(s) = (-\phi(s))(-\phi(s))$, then, by criterion 2 from [4], p. 507, we conclude that $(-\phi(s))^{-1}$ is CM. Then $\phi^{-2}(s)$ as a product of two CM functions is also CM (see [4], p. 507). Therefore, $\Diamond'(s) = (-\phi'(s))\phi^{-2}(s)$ is CM.

Subordination.

a) From the definition of $\nabla(s)$ it follows $e^{-t\nabla(s)} = e^{-t(s+\nabla(s))^{1/\gamma}}$. As $\nabla'(s)$ is CM, then $(s+\nabla(s))'$ is also CM. Then $e^{-t(s+\nabla(s))}$ is a LST for DF $E_t(x) * F_\nabla(x,t)$, where $E_t(x) = \begin{cases} 1, & x>t, \\ 0, & x \le t, \end{cases}$ and $*$ is the convolution sign.

It is known, that DF $U_t(x) = \int_0^\infty E_u(x) * F_\nabla(x,u) d_u G_{1/\gamma}(ut^{-\gamma})$ has a LST $e^{-t(s+\nabla(s))^{1/\gamma}}$ (see [4], p. 508). From the Uniqueness Theorem (see [4]) and from $e^{-t\nabla(s)} = e^{-t(s+\nabla(s))^{1/\gamma}}$ we obtain equality $F_\nabla(x,t) = \int_0^\infty E_u(x) * F_\nabla(x,u) d_u G_{1/\gamma}(ut^{-\gamma})$, which due to the relation $E_u(x) * F_\nabla(x,t) = F_\nabla(x-u,t)$ takes the form $F_\nabla(x,t) = \int_0^x F_\nabla(x-u,u) d_u G_{1/\gamma}(ut^{-\gamma})$, or in terms of densities –

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[1] Proof of Lemma 1is obviously and in case of function Δ is proofed in [3].

[2] From Theorem 2 it follows (see [4], p. 516) that $F_\Diamond(x,t)$ is infinitely divisible, and $\Diamond'(s) = \int_0^\infty e^{-sx} dP(x)$, where P is the measure on $[0,\infty)$.

$$f_{\nabla}(x,t)=\frac{1}{t^{\gamma}}\int_{0}^{x} f_{\nabla}(x-u,u)\,p\left(\frac{u}{t^{\gamma}},\frac{1}{\gamma},-\frac{1}{\gamma}\right)du .$$

b) Since $\Diamond(s)$ is the solution of equation $z^{\gamma}\mp z=s$, then the relation $\nabla(s)^{\gamma}+\nabla(s)=s+2\nabla(s)$, $s\geq 0$, holds, whence the formula $\nabla(s)=\Delta(s+2\nabla(s))$, $s\geq 0$, follows. As $\nabla'(s)$ is CM, then $(s+2\nabla(s))'$ is also CM. The function $e^{-t(s+2\nabla(s))}$ is a LST for DF $E_t(x)*F_{\nabla}(x,2t)$. Therefore, DF $U_t(x)=\int_0^{\infty}E_u(x)*F_{\nabla}(x,2u)d_uF_{\Delta}(u,t)$ has a LST $e^{-t\Delta(s+2\nabla(s))}$ (see [4], p. 508).

Now from $e^{-t\nabla(s)}=e^{-t\Delta(s+2\nabla(s))}$ and from the Uniqueness Theorem we obtain $F_{\nabla}(x,t)=\int_0^{\infty}E_u(x)*F_{\nabla}(x,2u)d_uF_{\Delta}(u,t)$, which due to the relation $E_u(x)*F_{\nabla}(x,2u)=F_{\nabla}(x-u,2u)$ takes the form $F_{\nabla}(x,t)=\int_0^{x}F_{\nabla}(x-u,2u)d_uF_{\Delta}(u,t)$, or in terms of densities – $f_{\nabla}(x,t)=\int_0^{x}f_{\nabla}(x-u,2u)f_{\Delta}(u,t)du$.

Maximal Likelihood Estimate of Parameter $\theta=t$ for $\gamma=2$. Let $\gamma=2$ and $X=(X_1,X_2,...,X_n)>0$ be a sample from general collection with DF $F_{\Diamond}(x,\theta)$, $\theta\in\Theta=(0,+\infty)$. Let's consider the likelihood function $L(X,\theta)=\prod_{k=1}^{n}f_{\Diamond}(X_k,\theta)$ with $f_{\Diamond}(x,\theta)=\theta\,\mathrm{e}^{-(\theta\pm x)^2/4x}\big/2x\sqrt{\pi x}$, and the likelihood equation

$$\frac{\partial\ln L(X,\theta)}{\partial\theta}=\sum_{k=1}^{n}\frac{1}{f_{\Diamond}(X_k,\theta)}\cdot\frac{\partial\ln f_{\Diamond}(X_k,\theta)}{\partial\theta}=0,$$

which takes the form $n\left[\frac{1}{\theta}\mp\frac{1}{2}\right]-\frac{\theta}{2}\sum_{k=1}^{n}\frac{1}{X_k}=0$. Denoting $p_n=(2n)^{-1}\sum_{k=1}^{n}X_k^{-1}$, we obtain the equation $p_n\theta^2\pm\theta/2-1=0$, the solution of which is $\hat{\theta}=\left(\mp1\pm\sqrt{1+16p_n}\right)\big/4p_n$. As $\hat{\theta}=\left(-1-\sqrt{1+16p_n}\right)\big/4p_n\notin\Theta$, then in case of ∇ for parameter θ we have just one estimate $\hat{\theta}=\left(-1+\sqrt{1+16p_n}\right)\big/4p_n\in\Theta$.

Connections with Other Laws. The density $f_{\Diamond}(x,\tau)$ and the DF $F_{\Diamond}(x,\tau)$ may be connected with functions $\Phi_{\beta,\mu}(s)=\frac{\beta}{2\pi i}\int_{\Gamma(\varepsilon,\beta)}e^{z^{\beta}-sz}z^{\beta(1-\mu)}dz$ and $\Psi_{\beta,\mu}(x)=\int_0^{x}\Phi_{\beta,\mu}(t)dt$, $x\in[0,\infty)$, $\beta\in(0,\pi)$, $1/\beta\leq\mu<+\infty$ (see [5]). Here for any $\varepsilon>0$ the contour $\Gamma(\varepsilon,\beta)$ of complex plane z is run in direction of decrease of $\arg(z)$. The contour consists of two rays (see Fig.1)

$L^{(\pm)}(\varepsilon,\beta)=\{z:\arg(z)=\pm\beta,\varepsilon\le|z|<+\infty\}$ and the arc of circle $l(\varepsilon,\beta)=$ $=\{z:|\arg(z)|\le\beta,|z|=\varepsilon\}$, joining the ends $\varepsilon e^{\pm i\beta}$ of those rays. It is proved in [5], that the function $\Phi_{\beta,\mu}(s)$ is entire and may be expanded into series

$$\Phi_{\beta,\mu}(s)=-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma(1-\mu+(k+1)\beta^{-1})}{k!}\sin\pi((k+1)\ /\beta-\mu)s^k\ . \qquad (2)$$

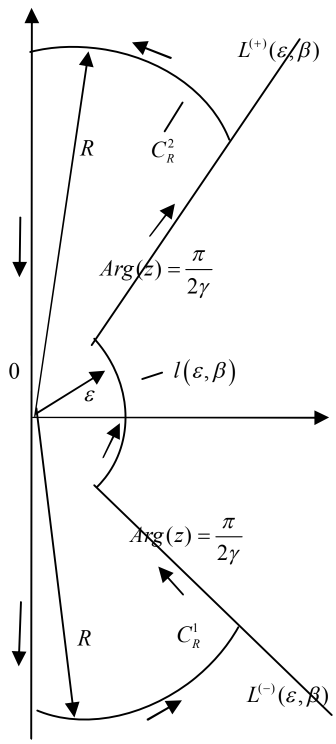

Fig. 1.

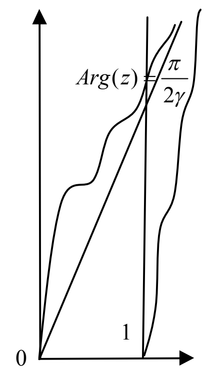

Fig. 2.

It is also known that $E_\beta(-x,\mu)=\int_0^{\infty}e^{-x\tau}d\Psi_{\beta,\mu}(\tau)$, $x\in[0,+\infty)$, $1\le\beta<+\infty$, $1/\beta\le\mu<+\infty$, where the entire functions $E_\beta(-x,\mu)$ of Mittag–Leffler type are defined through expansion

$$E_\beta(z,\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k\beta^{-1})},\ \beta>0\ ,\ -\infty<\mu<+\infty\ .$$

Taking $\mu=1$, $\beta=\gamma$ and $s=(\tau\pm x)x^{-1/\gamma}$ in formula (2), we obtain

$$f_\Diamond(x,\tau)=\tau\gamma^{-1}x^{1/\gamma-1}\Phi_{\gamma,1}((\tau\pm x)x^{-1/\gamma})\ .$$

Hence, from properties of $\Diamond$ and from relation $\Diamond(it)=\overline{\Diamond(-it)}$ we conclude, that $z=\nabla(it)\in\{\mathrm{Im}(z)\ge0,\ \mathrm{Re}(z)\ge1,\ 0\le\mathrm{Arg}(z)<\pi/2\gamma\}$ and $z=\Delta(it)\in\{\mathrm{Im}(z)\ge0,\ \mathrm{Re}(z)\ge1,\ 0\le\mathrm{Arg}(z)<\pi/2\gamma\}$ for $t\in(-\infty,0]$ (see Fig. 2). Then, due to conversion formula (see [4]) $f_\Diamond(x,\tau)=\frac{1}{2\pi}\mathrm{Re}\int_{-\infty}^{+\infty}e^{-itx-\tau\Diamond(-it)}dt=$ $=\frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{0}e^{-itx-\tau\Diamond(-it)}dt$, by methods from [1] and [3] it is easy to obtain the representation $f_\Diamond(x,\tau)=\mathrm{Re}\left\{\frac{1}{\pi i}\int_0^{i\infty}e^{xz^\gamma-(\tau\pm x)z}(\gamma z^{\gamma-1}\mp1)dz\right\}$. Comparing the latter with (1), we conclude $f_{\Diamond}(x,\tau)=\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xz^{\gamma}-(\tau\pm x)z}(\gamma z^{\gamma-1}\mp 1)dz\right\}$, where changing the integration variable $y=x^{1/\gamma}z$, we obtain

$$f_{\Diamond}(x,\tau)=\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{y^{\gamma}-(\tau\pm x)x^{-1/\gamma}y}(\gamma x^{1/\gamma-1}y^{\gamma-1}\mp 1)x^{-1/\gamma}dy\right\}.$$

Let's consider a closed contour $\Upsilon=C_R^1\cup[-R,R]\cup C_R^2\cup\Gamma_R$, where $C_R^2=\{z=\mathrm{Re}^{i\varphi},\beta\le\varphi\le\pi/2\gamma\}$, $C_R^1=\{z=Re^{i\varphi},-\pi/2\le\varphi\le-\beta\}$, $\pi/2\gamma<\beta<\pi/2$, and let Γ_R be the part of the curve Γ (see Fig. 1), bounded by those arcs. As $h(y)=e^{y^{\gamma}-(\tau\pm x)x^{-1/\gamma}y}\left(\gamma x^{1/\gamma-1}y^{\gamma-1}\mp 1\right)x^{-1/\gamma}$ is analytical inside of Υ, then, due to Cauchy Theorem, we get $\int_{\Upsilon}h(y)dy=0$. Let's show that $\int_{C_R^1,C_R^2}\xrightarrow[R\to+\infty]{}0$.

Changing the integration variable $y=\mathrm{Re}^{i\varphi}$ and taking into account that $\cos\gamma\varphi\le 0$, we get

$$\left|\int_{C_R^1,C_R^2} e^{y^{\gamma}-(\tau\pm x)x^{-1/\gamma}y}\,\frac{\gamma x^{1/\gamma-1}y^{\gamma-1}\mp 1}{x^{1/\gamma}}dy\right|\le\left|\int_{\beta}^{\pi/2} e^{R^{\gamma}\cos\gamma\varphi-(\tau\pm x)x^{-1/\gamma}R\cos\varphi}\,\frac{\gamma x^{1/\gamma-1}R^{\gamma-1}+1}{x^{1/\gamma}}Rd\varphi\right|\to 0.$$

Tending R to infinity, we obtain

$$f_{\Diamond}(x,\tau)=\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{\Gamma(\varepsilon,\beta)} e^{y^{\gamma}-(\tau\pm x)x^{-1/\gamma}y}\left(\gamma x^{1/\gamma-1}y^{\gamma-1}\mp 1\right)x^{-1/\gamma}dy\right\},$$

and comparing the latter representation with $\Phi_{\beta,\mu}(s)$, we get

$$f_{\Diamond}(x,\tau)=x^{-1}\Phi_{\gamma,1/\gamma}\left((\tau\pm x)x^{-1/\gamma}\right)\mp\gamma^{-1}x^{-1/\gamma}\Phi_{\gamma,1}\left((\tau\pm x)x^{-1/\gamma}\right).$$

Taking into account the relation $f_{\Diamond}(x,\tau)=\tau\gamma^{-1}x^{1/\gamma-1}\Phi_{\gamma,1}\left((\tau\pm x)x^{-1/\gamma}\right)$, we get

$$f_{\Diamond}(x,\tau)=\tau x^{-1}\left(\tau\pm x^{1-2/\gamma}\right)^{-1}\Phi_{\gamma,1/\gamma}\left((\tau\pm x)x^{-1/\gamma}\right).$$

Let $0<\gamma_1<\gamma_2<+\infty$, $0<\mu_1<+\infty$ and $-\infty<\mu_2<+\infty$. Then (see [5]),

$$E_{\gamma_2}(z,\mu_2)=\int_0^{\infty}E_{\gamma_1}\left(z\tau^{1/\gamma_1},\mu_1\right)\tau^{\mu_1-1}\Phi_{\gamma,\mu}(\tau)d\tau,\ |z|<+\infty,\ \mu=\mu_2+\frac{\gamma_1}{\gamma_2}(1-\mu_1)\ \gamma=\frac{\gamma_2}{\gamma_1}. \quad (3)$$

Let X and Y be independent RVs with DFs F and G, and with LSTs φ and ϕ respectively. It is known that (see [4], p. 521) the LST for the RV XY is given by the Parseval's equality

$$\int_0^{\infty}\varphi(sx)dG(x)=\int_0^{\infty}\phi(sx)dF(x).$$

Therefore, relation (3) is the LST for the RV XY, where RVs X and Y are independent and have DFs $\Psi_{\gamma_1,\mu_1}(x)$ and $G(x)$ respectively. Here $dG(x)=x^{\gamma_1(\mu_1-1)}d\Psi_{\gamma,\mu}(x^{\gamma_1})$ (i. e. $G(x)$ has density $\gamma_1 x^{\gamma_1(\mu_1-2)-1}\Phi_{\gamma,\mu}(x^{\gamma_1})$). Or, as

$E_{\gamma_1}\left(z\tau^{1/\gamma_1},\mu_1\right)=\int\limits_0^\infty e^{zv}\tau^{-1/\gamma_1}\Phi_{\gamma_1,\mu_1}\left(v\tau^{-1/\gamma_1}\right)dv$, we can consider the relation (3) as the LST of RV XY, where RVs X and Y have densities $\tau^{-1/\gamma_1}\Phi_{\gamma_1,\mu_1}\left(v\tau^{-1/\gamma_1}\right)$ and $\tau^{\mu_1-1}\Phi_{\gamma,\mu}(\tau)$ respectively.

Let's consider stable densities $p(x,\alpha,\varphi)$ (see [4], p. 657) with characteristic function (CF) $g(t)=\exp\left\{-|t|^\alpha\exp\{\pm\pi i\varphi/2\}\right\}$, where $0<\alpha<2$, $\pm=\operatorname{sign}(t)$, $|\varphi|\le\begin{cases}\alpha, & 0<\alpha<1,\\ 2-\alpha, & 1<\alpha<2.\end{cases}$ One-sided densities $p(x,1/\gamma,-1/\gamma)$, $1<\gamma\le 2$, are of special interest here.

In Fig. 3 the graphs[1] of densities $p(x,1/\gamma,-1/\gamma)$ and $f_\Diamond(x,1)$ for $\gamma=1{,}5;\ 1{,}6;\ 1{,}7;\ 1{,}8;\ 2$ are presented. In Fig. 4 the graphs of density $f_\Diamond(x,1)$ for large values of the argument and those close to zero in the case of $\gamma=1{,}7$ are presented. Oscillation in the graphs confirms the need to consider the asymptotical behavior of those densities on infinity and nearly zero separately.

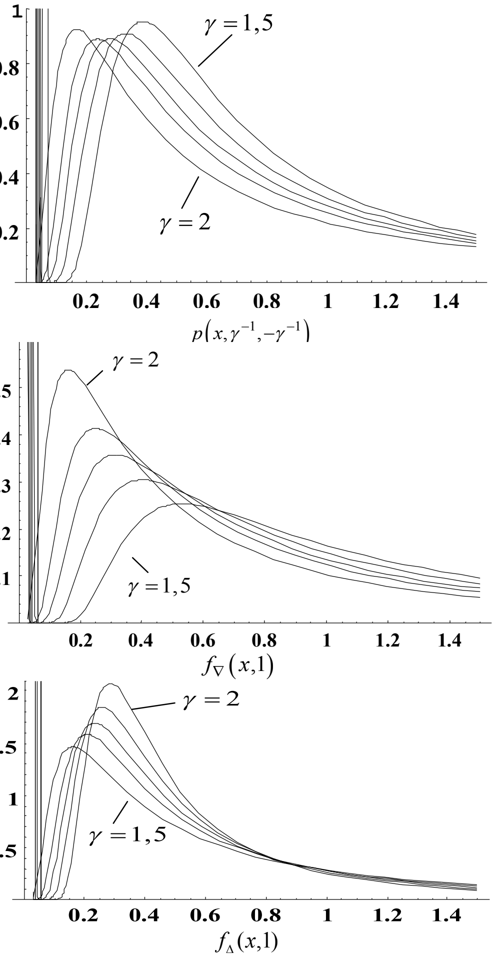

Fig. 3.

Substituting $k=n+1$ in (1) and taking into account the formula $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, we obtain

$$f_\Diamond(x,t)=\frac{t}{\pi}\sum_{k=1}^\infty\frac{(-1)^{k-1}(t\pm x)^{k-1}}{k!}\Gamma(k\gamma^{-1}+1)x^{-k\gamma^{-1}-1}\sin\frac{\pi k}{\gamma},\ x>0,\ 0<t<\infty.$$

Since $0<1/\gamma<1$, then from [4] (p. 659, Lemma 1), for function ∇ we have $f_\nabla(x,t)=t(t+x)^{-(\gamma+1)}p\left(x/(t+x)^\gamma,\gamma^{-1},-\gamma^{-1}\right)$. For function Δ and $0<x<t<\infty$ we obtain $f_\Delta(x,t)=t(t-x)^{-\gamma-1}p(x/(t-x)^\gamma,1/\gamma,-1/\gamma)$, and for $0<t<x<\infty$ we get $f_\Delta(x,t)=t(x-t)^{-\gamma-1}p(x/(x-t)^\gamma,1/\gamma,3/\gamma-2)$.

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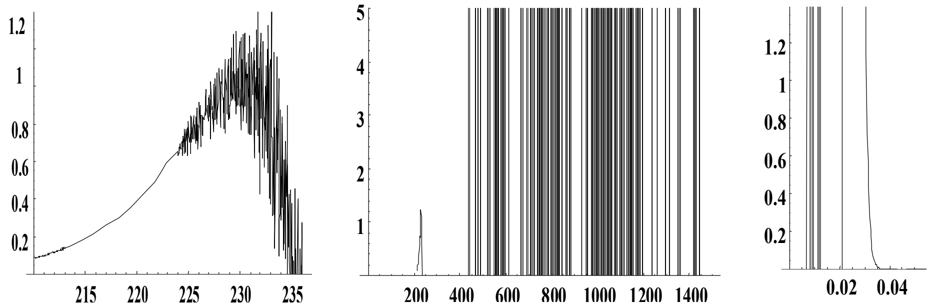
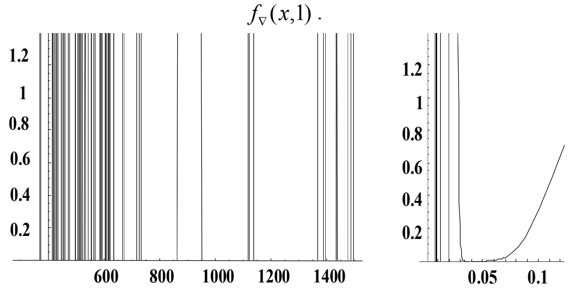
[1] The graphs are constructed by means of program Matematika, whereas the first 2500 members of corresponding series are taken.



$f_\nabla(x,1)$.



$f_\Delta(x,1)$.

Fig. 4.

Due to formula $\Gamma(\alpha)\Gamma(1-\alpha)=\pi/\sin\pi\alpha$, $0<\alpha<1$, we obtain $f_\Delta(t,t)=x^{-1/\gamma}\big/\gamma\Gamma\left((\gamma-1)\gamma^{-1}\right)$.

For density $f_{1/\gamma}(x,t)$, $x,t>0$, with LST $e^{-ts^{1/\gamma}}$ we have (see [4], p. 659, Lemma 1)

$$f_{1/\gamma}(x,t)=\begin{cases}\dfrac{t}{\gamma\pi x}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\Gamma\left(\frac{n+1}{\gamma}\right)x^{-\frac{n+1}{\gamma}}\sin\frac{n+1}{\gamma}\pi, & 1<\gamma<2,\\ \left(t\big/\left(2x\sqrt{\pi x}\right)\right)\exp\left\{-t^2/4x\right\}, & \gamma=2,\end{cases}$$

and, denoting $\delta(\tau)=\mp\tau$ for $\Diamond(s)$ and $\delta(\tau)=0$ for $s^{1/\gamma}$, we obtain

$$f_{\Diamond,1/\gamma}(x,t)=\begin{cases}\dfrac{t}{\gamma\pi x}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n (t-\delta(x))^n}{n!}\Gamma\left(\frac{n+1}{\gamma}\right)x^{-\frac{n+1}{\gamma}}\sin\frac{n+1}{\gamma}\pi, & 1<\gamma<2,\\ \left(t\big/\left(2x\sqrt{\pi x}\right)\right)\exp\left\{-\left(t-\delta(x)\right)^2/4x\right\}, & \gamma=2,\end{cases}$$

for densities $f_\Diamond(x,t)$ and $f_{1/\gamma}(x,t)$. From here we finally get $(t-\delta(x))f_\Diamond(x,t)=tf_{1/\gamma}(x,t-\delta(x))$ (for Δ it holds only in case of $x<t$).

Let's consider the function $\Diamond^{-1}(s)=b(s)=s^\gamma+\delta(s)$. As the CF $e^{(-it)^\gamma}=e^{-|t|^\gamma e^{\pm\frac{\pi i(2-\gamma)}{2}}}$ has density $p(x,\gamma,2-\gamma)$, it is easy to see that CF $e^{\tau b(-it)}$ has density $\varphi\left(x,\tau\right)=\tau^{-1/\gamma}p((x+\delta(\tau))\tau^{-1/\gamma},\gamma,2-\gamma)$. Then since

$$\varphi(-x,\tau)=\frac{1}{\pi\gamma}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\delta(\tau))^n}{n!}\Gamma\left(\frac{n+1}{\gamma}\right)\tau^{-\frac{n+1}{\gamma}}\sin\frac{\pi(n+1)}{\gamma},$$

we have $tf_\Diamond(t,x)=x\varphi_\Diamond(-x,t)$.

Taking into account that $p(-x,\alpha,\varphi)=p(x,\alpha,-\varphi)$, from [4] (p. 659, Lemma 1) we get

$$\frac{1}{\tau^{1/\gamma}}p\left(\frac{\pm\tau-x}{\tau^{1/\gamma}},\gamma,2-\gamma\right)=\sum_{n=0}^{\infty}\frac{(-1)^n(x\mp\tau)^n}{\pi\gamma n!}\Gamma\left(\frac{n+1}{\gamma}\right)\tau^{-\frac{n+1}{\gamma}}\sin\frac{\pi(n+1)}{\gamma}=\frac{\tau f_\Diamond(\tau,x)}{x}.$$

We formulate the above obtained results in the following lemma.

Lemma 2. The following relations are true:

$$f_\nabla(x,t)=t(t+x)^{-\gamma-1}p(x(t+x)^{-\gamma},\gamma^{-1},-\gamma^{-1}),\ x>0,\ 0<t<\infty;$$

$$f_\Delta(x,t)=\begin{cases}t(t-x)^{-\gamma-1}p(x(t-x)^{-\gamma},\gamma^{-1},-\gamma^{-1}), & 0<x<t<\infty,\\ t(x-t)^{-\gamma-1}p(x(x-t)^{-\gamma},\gamma^{-1},3\gamma^{-1}-2), & 0<t<x<\infty,\\ \left[\gamma\Gamma((\gamma-1)\gamma^{-1})\right]^{-1}x^{1/\gamma}, & x=t>0;\end{cases}\qquad(4)$$

$$f_\Diamond(\tau,x)=x\tau^{-1-1/\gamma}p((\pm\tau-x)\tau^{-1/\gamma},\gamma,2-\gamma),\ x>0,\ \tau>0;$$

$$tf_\Diamond(t,x)=x\varphi_\Diamond(-x,t),\ (t-\delta(x))f_\Diamond(x,t)=tf_{1/\gamma}(x,t-\delta(x))\ [1].$$

Asymptotical expansions for $f_\Diamond(t,x)$ can directly be obtained from Lemma 2 and from the corresponding expansions of densities $p(x,\alpha,\varphi)$ (see [6]).

Let $a_n=a_n(\alpha)=\dfrac{2^{2n}|B_{2n}|}{2n(2n)!}\left[\alpha(1-\alpha)^{2n-1}+1-(1-\alpha)^{2n}\right]$ be polynomials of degree $2n-1$ with absolute term equal to zero (B_n are Bernoulli numbers) and $C_n(y_1,y_2,...,y_n)=\sum\limits_{k_1+2k_2+...+nk_n=n,k_j\ge0}\dfrac{n!}{k_1!...k_n!}\left(\dfrac{y_1}{1!}\right)^{k_1}...\left(\dfrac{y_n}{n!}\right)^{k_n}$ be Bell polynomials.

Denote:

$b_n(\alpha)=C_n(1!a_1,2!a_2,...,n!a_n)/n!$, $d_n=d_n(\vartheta,\alpha)=(a_n(\alpha)-\vartheta^2\alpha^{-1}b_{n+1}(\alpha))\vartheta^{2n}$, $q_n=q_n(\vartheta,\alpha)=C_n(1!d_1,2!d_2,...,n!d_n)/n!$ (polynomials of degree $2(n+1)$ by ϑ and $2n$ by α). Let $\xi=\xi(x,\alpha)=(1-\alpha)(x/\alpha)^{\alpha/(\alpha-1)}$ and $\nu=\nu(\alpha)=(1-\alpha)^{-1/\alpha}$.

Theorem 3 [2]**.** For $f_\Diamond(t,x)$ the following expansions are true:

1. $f_\nabla(x,t)\sim t\nu\,\mathrm{e}^{-\xi}(t+x)^{-\gamma-1}\sqrt{\dfrac{\gamma}{2\pi}}\xi^{\frac{2\gamma-1}{2}}\left(1+\sum\limits_{n=1}^{\infty}Q_n(\gamma^{-1})(\gamma\xi^{-1})^n\right)$, when $\dfrac{x^{1/\gamma}}{t-x}\to0$,

where $\xi=\xi(x/(t+x)^\gamma,\gamma^{-1})=(\gamma-1)\gamma^{-1}(\gamma x/(t+x)^\gamma)^{1/(1-\gamma)}$, $\nu=\nu(1/\gamma)=(\gamma/(\gamma-1))^\gamma$.

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[1] The latter relation for Δ is true only for $x<t$.

[2] From Theorem 3 the asymptotical expansions follow for $f_\Diamond(t,x)$ for cases when $t=const$ and $x\to0$, $t=const$ and $x\to\infty$, $x=const$ and $t\to\infty$, $x+t\to\infty$, $x+t\to0$ itc.

2. $f_\Delta(x,t) \sim t\nu\, \mathrm{e}^{-\xi}(t-x)^{-\gamma-1}\sqrt{\dfrac{\gamma}{2\pi}}\,\xi^{\frac{2\gamma-1}{2}}\left(1+\sum_{n=1}^{\infty}Q_n(\gamma^{-1})(\gamma\xi^{-1})^n\right)$, when $x^{1/\gamma}(t-x)^{-1}\downarrow 0$, where $\xi=\xi\left(x/(t-x)^\gamma,\gamma^{-1}\right)=(\gamma-1)\gamma^{-1}\left(\gamma x/(t-x)^\gamma\right)^{1/(1-\gamma)}$, $\nu=\nu(\gamma^{-1})=\left(\gamma/(\gamma-1)\right)^\gamma$.

3. $f_\Delta(x,t) \sim \dfrac{\gamma t}{\pi}\sum_{k=1}^{\infty}\dfrac{\Gamma(\gamma k)x^{k-1}}{(k-1)!(x-t)^{\gamma k+1}}\sin(\pi k(2-\gamma))$, when $\dfrac{x^{1/\gamma}}{t-x}\uparrow 0$ and $1<\gamma<2$.

4. $f_\Delta(x,t) \sim \dfrac{t}{\sqrt{\pi x}}\mathrm{e}^{-(x-t)^2/4x}\left(1+\sum_{n=1}^{\infty}Q_n\left(\dfrac{1}{2}\right)(8x(x-t)^{-2})^n\right)$, when $\dfrac{x^{1/\gamma}}{t-x}\uparrow 0$ and $\gamma=2$.

Here $Q_n=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}q_n(t)e^{-t^2/2}dt$, where q_n are the above-defined polynomials of degree $2n$ with respect to α.

Theorem 4 (Unimodality). DF $F_\Diamond(t,x)$ is unimodal.

The proof follows from (4). As the density $p(x,\gamma,\gamma-2)$ is unimodal (see [6]), then $f_\Diamond(t,x)$ is also unimodal (since x does not break unimodality)[1].

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REFERENCES

1. **Martirosyan A.R.** Asymptotical Analysis of Characteristics of Insurance Models in Critical Situations. Yer.: Diss. for Scientific Degree of Candidate of Physical and Mathematical Sciences, 2008 (http://www.prorector.org/avtors/martirosyan/diss.pdf) (in Russian).
2. **Sahakyan V.G.** The Discipline of Random Choice in Models $\vec{M}_r\,|\,\vec{G}_r\,|1|\,\infty$. M.: Diss. for Scientific Degree of Candidate of Physical and Mathematical Sciences, 1985 (in Russian).
3. **Chitchyan R.N., Ugarid M.** On Procedure of Conversion of LST of Limit Distributions in the Model $\vec{M}_r\,|\,\vec{G}_r\,|1|\,\infty$. VC AN Arm. SSR, 1990, preprint (in Russian).
4. **Feller V.** Introduction to Probability Theory and Its Application. V. 2. M.: Mir, 1984.
5. **Djrbashyan M.M., Baghyan R.A.** Izv. AN Arm. SSR. Matematika, 1975, v. X, № 6, p. 482–508 (in Russian).
6. **Zolotarev V.M.** One-Dimensional Stable Distributions. M.: Nauka, 1980.

[1] The density $p(x,\gamma,\gamma-2)$ has the mode m_γ (see [6]). Therefore, $f_\Diamond(t,x)$ has the mode $m_\Diamond=m_\gamma\tau^{1/\gamma}\pm\tau$.

Ա. Ռ. Մարտիրոսյան

Ռիսկերի տեսության մի սահմանային բաշխման հատկությունները

Ուսումնասիրված են ռիսկերի տեսության որոշ սահմանային թեորեմներում առաջացող մի պատահական պրոցեսի հատկությունները: Տրված են կայուն բաշխումների և ֆունկցիաների մի դասի հետ ունեցած կապերը և ուսումնասիրված է այդ բաշխման ասիմպտոտիկ վարքը:

А. Р. Мартиросян.

Свойства одного предельного закона теории риска

В работе исследованы свойства одного случайного процесса, возникающего во многих предельных теоремах теории риска. Даны связи с устойчивыми законами и с одним классом интегральных преобразований. Исследованы асимптотические свойства этого закона.