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ON INDEPENDENCE NUMBER OF STRONG GENERALIZED CYCLES PRODUCT

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In the present paper the independence number of generalized cycles product is investigated. A method for constructing the maximal independent set in the product graph is presented. The method is particularly based on a specific combinatorial problem, which is also solved in the paper. The main result generalizes the similar fact known for odd cycles [6].

Keywords: independence number, cycles product, generalized cycles.

Introduction. Investigation of independence number (IN) of graphs product comes from an information theory problem, examined by Shannon in [1, 2]. A problem was raised in [3, 4]: to find necessary and sufficient conditions on a finite graph G for the IN to be multiplicative on the product $G \times H$ for every finite graph H. The sufficient condition was found by Shannon [1]. Later Rosenfeld [5] proved that this condition was not necessary and gave the necessary and sufficient condition, thereby introducing an invariant ρ – the Rosenfeld number. In [6] Hales introduced a method of finding the IN of odd cycles product. Later, in [7] the lower estimate for the IN the generalized cycles product was obtained. In this paper we give a method to construct the maximal independent set of generalized cycles product. For other references on the IN, graphs product and its applications see [3, 8, 9]. In [10] the IN of the of generalized cycles direct product is found.

Essential Notations. The set of a graph vertices is independent, if no two vertices in it are adjacent. An independence set containing k vertices is called k-independent set. Let's denote by $\alpha(G)$ the number of vertices in the maximal independent set of G. A graph is called k-regular, if the degree of each vertex is k.

We say, that C_n^k is a generalized cycle, iff it is a $2k$-regular graph with n vertices, which can be ordered on a circumference so that each vertex is adjacent to its k preceding and succeeding vertices ($n > 2$, $1 \le k \le [(n-1)/2]$), where $[c]$ is the greatest integer less than or equal to c.

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The strong product of G_1 and G_2 is a graph G with vertices $V(G)$ and edges $E(G)$, where $V(G)=V(G_1)\times V(G_2)$ and $[(u_1,u_2),(v_1,v_2)]\in E(G)$, iff $u_1\to v_1$ and $u_2\to v_2$ ($u\to v$ means either $u=v$ or $[u,v]$ is an edge in the appropriate graph).

A non-negative real-valued function f on $V(G)$ is called admissible, if for each clique (the maximal set of pairwise adjacent vertices) C holds $\sum_{v\in C} f(v)\le 1$. The Rosenfeld number $\rho(G)$ of G is defined as $\rho(G)=\max_f \sum_{v\in V(G)} f(v)$, running over all admissible functions f [5, 6].

One can deduce $\rho(C_n^k)=n/(k+1)$ and $\alpha(C_n^k)=[\rho(C_n^k)]$. The following inequalities are known for arbitrary graphs G and H [5, 6]:

$$\alpha(G)\times\alpha(H)\le\alpha(G\times H)\le\min(\rho(G)\times\alpha(H),\alpha(G)\times\rho(H)). \quad (1)$$

Hales [6] obtained the following result for the IN of the product of two odd cycles:

$$\alpha(C_{2n+1}\times C_{2k+1})=\min([\rho(C_{2n+1})\times\alpha(C_{2k+1})],\ [\alpha(C_{2n+1})\times\rho(C_{2k+1})]).$$

Our result generalizes the equality above for generalized cycles

$$\alpha(C_m^p\times C_n^k)=\min([\rho(C_m^p)\times\alpha(C_n^k)],\ [\alpha(C_m^p)\times\rho(C_n^k)]).$$

Results and Discussions. The main difference between the method we are going to introduce and the one in [7] is that we'll make use of the optimization problem below to achieve the known upper bound (1). Thus, let's consider the following optimization problem and denote it by $S(m,p,\alpha)$:

$$\begin{cases} x_1+x_2+\ldots+x_p\le\alpha, \\ x_2+x_3+\ldots+x_{p+1}\le\alpha, \\ \ldots \\ x_m+x_1+\ldots+x_{p-1}\le\alpha, \end{cases} \quad (2)$$

$\sum_{i=1}^m x_i\to\max$, where $p,m,\alpha\in N\cup\{0\}$; $i=1,2,\ldots,m$; $p\le m$. If we sum all the lines of (2), we'll have $p\sum_{i=1}^m x_i\le m\alpha$ or $\sum_{i=1}^m x_i\le[m\alpha/p]$. We'll construct a solution with value $[m\alpha/p]$ and satisfying the condition $\max_{i,j=1,2\ldots,m}(x_i-x_j)\le 1$. In that case, after denoting $t=\min_{i=1,2,\ldots,m} x_i$, we can consider 0–1 optimization problem instead, i.e. $x_i\in\{0,1\}$, $i=1,2,\ldots,m$; $\alpha\le p\le m$. Obviously, if $r_{mp}=m(\mathrm{mod}\,p)=0$, the solution for (2) is: $(\underbrace{\underbrace{1,1,\ldots,1}_{\alpha},0,0,\ldots,0}_{p},\underbrace{\underbrace{1,1,\ldots,1}_{\alpha},0,0,\ldots,0}_{p},\ldots,\underbrace{\underbrace{1,1,\ldots,1}_{\alpha},0,0,\ldots,0}_{p})$. Thus, let's consider the case when $r_{mp}\ne 0$ and divide the vector of unknowns in the following way:

$$(\underbrace{x_1,x_2,\ldots,x_p}_{p},\underbrace{x_{p+1},x_{p+2},\ldots,x_{2p}}_{p},\ldots,\underbrace{x_{([m/p]-1)p+1},x_{([m/p]-1)p+2},\ldots,x_{[m/p]p}}_{p},\underbrace{x_{[m/p]p+1},\ldots,x_m}_{r_{mp}}).$$

We assume that: 1. $\sum_{i=1}^{p} x_i = \alpha$; 2. $x_i = x_{i+p} = ... = x_{i+([m/p]-1)p}$, $i = 1,2,...,p$.

Hence, the vector of unknowns may be rewritten as follows

$$(\underbrace{\underbrace{x_1,x_2,...,x_p}_{p},\underbrace{x_1,x_2,...,x_p}_{p},...,\underbrace{x_1,x_2,...,x_p}_{p}}_{[m/p]},\underbrace{x_{p+1},...,x_{p+r_{mp}}}_{r_{mp}}) .$$

We'll determine the values of $x_1, x_2,..., x_{p+r_{mp}}$ so that the vector satisfies inequalities (2) and the following equalities

$$\sum_{i=1}^{p} x_i = \alpha, \quad \sum_{i=p+1}^{p+r_{mp}} x_i = [m\alpha / p] - [m/p]\alpha . \qquad (3)$$

Note that $[m\alpha / p] - [m/p]\alpha = [m/p]\alpha + [r_{mp}\alpha / p] - [m/p]\alpha = [r_{mp}\alpha / p] \le r_{mp}$.

Since $\sum_{i=1}^{p} x_i = \alpha$ the conditions (2) are equivalent to the following inequalities:

$$\begin{cases} \sum_{i=p+1}^{p+k} x_i \le \sum_{i=1}^{k} x_i, & k = 1,...,r_{mp}, \\ \sum_{i=p+r_{mp}-k}^{p+r_{mp}} x_i \le \sum_{i=p-k}^{p} x_i, & k = 0,...,r_{mp}-1, \\ \sum_{i=p+1}^{p+r_{mp}} x_i \le \sum_{i=k}^{k+r_{mp}-1} x_i, & k = 1,..., p - r_{mp} + 1. \end{cases} \qquad (4)$$

Thus the problem can be formulated as follows: find the 0–1 unknowns $(x_1, x_2,..., x_{p+r_{mp}})$, satisfying equalities (3) and inequalities (4). We'll show that there exists a vector, satisfying those conditions. Let $r < p$. For given $\alpha, \beta \in N$, $\alpha \le p$, $\beta \le r$, the 0–1 vector $(x_1, x_2,..., x_p, x_{p+1}..., x_{p+r})$ is admissible, iff

$$\sum_{i=1}^{p} x_i = \alpha \le p, \quad \alpha \in N; \quad \sum_{i=p+1}^{p+r} x_i = \beta \le r, \quad \beta \in N. \qquad (*)$$

Consider the following two problems:

1. Let $\beta / r \le \alpha / p$, determine admissible 0–1 vector $(x_1, x_2,..., x_p, x_{p+1}..., x_{p+r})$, so that

$$\begin{cases} \sum_{i=p+1}^{p+k} x_i \le \sum_{i=1}^{k} x_i, & k = 1,...,r, \\ \sum_{i=p+r-k}^{p+r} x_i \le \sum_{i=p-k}^{p} x_i, & k = 0,...,r-1, \\ \sum_{i=p+1}^{p+r} x_i \le \sum_{i=k}^{k+r-1} x_i, & k = 1,..., p - r + 1. \end{cases} \qquad (**)$$

2. Let $\beta / r \ge \alpha / p$, determine admissible 0–1 vector $(x_1, x_2,..., x_p, x_{p+1}..., x_{p+r})$, so that

$$\begin{cases} \sum_{i=p+1}^{p+k} x_i \ge \sum_{i=1}^{k} x_i, & k=1,...,r, \\ \sum_{i=p+r-k}^{p+r} x_i \ge \sum_{i=p-k}^{p} x_i, & k=0,...,r-1, \\ \sum_{i=p+1}^{p+r} x_i \ge \sum_{i=k}^{k+r-1} x_i, & k=1,...,p-r+1. \end{cases} \qquad (***)$$

We'll show by induction on r that both problems have solutions. If $r=1$, then the statement is true. Indeed, for the first case we have $\beta = \beta / r \le \alpha / p$, $\beta = 0,1$, and any 0–1 vector $(x_1, x_2, ..., x_p, x_{p+1})$, satisfying equalities (*), is a solution for the first problem. The second case can be deduced analogously.

Now assume that the statement is true for all natural numbers less than r. Consider the first case for r (the second one can be proved analogously). Let $r_1 = p(\mathrm{mod}\, r)$, if $\alpha - \beta[p/r] \ge r_1$, then

$$(\overbrace{\underbrace{x_1, x_2, ..., x_r}_{r}, \underbrace{x_1, x_2, ..., x_r}_{r}, ..., \underbrace{x_1, x_2, ..., x_r}_{r}, \underbrace{1,1,...,1}_{r_1}}^{p}, \underbrace{x_1, x_2, ..., x_r}_{r}) \text{ vector, where } \sum_{i=1}^{r} x_i = \beta,$$

satisfies all the conditions except perhaps the first equality in (*). By replacing the first zero components $\alpha - \beta[p/r] - r_1$ in the mentioned vector with ones we'll yield the solution. Thus, we can assume that $\alpha - \beta[p/r] < r_1$. Consider the vector

$$(\overbrace{\underbrace{x_1, x_2, ..., x_r}_{r}, \underbrace{x_1, x_2, ..., x_r}_{r}, ..., \underbrace{x_1, x_2, ..., x_r}_{r}, \underbrace{x_{r+1}, x_{r+2}, ..., x_{r+r_1}}_{r_1}}^{p}, \underbrace{x_1, x_2, ..., x_r}_{r}), \text{ where } \sum_{i=1}^{r} x_i = \beta$$

and $\sum_{i=r+1}^{r+r_1} x_i = \alpha - \beta[p/r]$. For this vector, obviously, equalities (*) are satisfied. Now let's rewrite inequalities (**) for the mentioned vector. It's easy to see that the first inequality is satisfied, while the second is equivalent to the following two inequalities: $\sum_{i=r-k}^{r} x_i \le \sum_{i=r+r_1-k}^{r+r_1} x_i$, $k = 0,...,r_1 - 1$;

$$\sum_{i=r-k}^{r-(k-r_1)-1} x_i + \sum_{i=r-(k-r_1)}^{r} x_i \le \sum_{i=r-(k-r_1)}^{r} x_i + \sum_{i=r+1}^{r+r_1} x_i, \; k = r_1, ..., r-1.$$

The third one is equivalent to $\sum_{i=1}^{r} x_i \le \sum_{i=k+1}^{r} x_i + \sum_{i=r+1}^{r+k} x_i$, $k = 1,...,r_1$. After reducing the inequalities we get

$$\begin{cases} \sum_{i=r+1}^{r+k} x_i \ge \sum_{i=1}^{k} x_i, & k=1,...,r_1, \\ \sum_{i=r+r_1-k}^{r+r_1} x_i \ge \sum_{i=r-k}^{r} x_i, & k=0,...,r_1-1, \\ \sum_{i=r+1}^{r+r_1} x_i \ge \sum_{i=r-k}^{r-(k-r_1)-1} x_i, & k=r_1,...,r-1, \end{cases}$$

where $\sum_{i=1}^{r} x_i = \beta$ and $\sum_{i=r+1}^{r+r_1} x_i = \alpha - \beta[p/r]$. Since $\beta / r \le \alpha / p$ and

$$\frac{\alpha-\beta[p/r]}{r_1}=\frac{\alpha-\beta(\frac{p}{r}-\frac{r_1}{r})}{r_1}=\frac{\alpha-\frac{\beta}{r}(p-r_1)}{r_1}\ge\frac{\alpha-\frac{\alpha}{p}(p-r_1)}{r_1}=\frac{\alpha}{r_1}\left(1-\frac{p-r_1}{p}\right)=\frac{\alpha}{p}\ge\frac{\beta}{r},$$

we obtained a 2nd type problem for $r_1 < r$. The statement is true by induction. Thus, we constructed an optimal solution for problem (2). Let r_{mp} and α_{mp} be acronyms for $m(\mathrm{mod}(p+1))$ and $\alpha(C_m^p)$, respectively, then the main theorem can be formulated.

Theorem. If $\rho(C_n^k)\alpha(C_m^p) \ge \alpha(C_n^k)\rho(C_m^p)$, then $\alpha(C_m^p \times C_n^k) \ge [\alpha(C_n^k)\rho(C_m^p)]$.

Proof. To prove the Theorem it's enough to construct an independent set in the product graph with the specified cardinality. We'll construct $t=\alpha_{mp}$, α_{nk}-independent sets $S_0,...,S_{t-1}$ in graph C_n^k, then decompose each of them into $p+1$ parts. Afterwards by constructing r_{mp} more independent sets in C_n^k, we'll get m independent sets $P_0,P_1,...,P_{m-1}$. Finally, we'll show that the following independent set in the product graph is the required one (vertices of generalized cycles are denoted by numbers in the above mentioned cyclical order):

$$S=\bigcup_{i=0}^{m-1} B_i, \quad B_i=\{(i,v) / v \in P_i\}.$$

Now, let's consider the problem $S(m,p+1,\alpha_{nk})$ and denote the above constructed optimal solution by

$$(\underbrace{\underbrace{x_0,x_1,...,x_p}_{p+1},\underbrace{x_0,x_1,...,x_p}_{p+1},...,\underbrace{x_0,x_1,...,x_p}_{p+1}}_{[m/(p+1)]},\underbrace{x_{p+1},...,x_{p+r_{mp}}}_{r_{mp}}).$$ It is clear that $\alpha_{nk}=\sum_{i=0}^{p} x_i$.

Suppose l is the least non-negative integer, satisfying inequality $(l+1)r_{nk} \ge (k+1)r_{mp}\alpha_{nk}/(p+1)$. According to the supposition of the Theorem

$$\rho(C_n^k)\alpha(C_m^p)=\alpha_{mp}\frac{n}{k+1}=\alpha_{mp}\alpha_{nk}+\alpha_{mp}\frac{r_{nk}}{k+1}\ge\alpha_{nk}\alpha_{mp}+\alpha_{nk}\frac{r_{mp}}{p+1}=\alpha_{nk}\frac{m}{p+1}=\alpha(C_n^k)\rho(C_m^p),$$

and, therefore, $l<\alpha_{mp}$. Consider the following α_{nk}-independent sets in the graph C_n^k

$$S_0=\{0,(k+1),2(k+1),...,(\alpha_{nk}-1)(k+1)\},$$
$$S_1=\{-r_{nk},(k+1)-r_{nk},2(k+1)-r_{nk},...,(\alpha_{nk}-1)(k+1)-r_{nk}\},$$
$$S_2=\{-2r_{nk},(k+1)-2r_{nk},2(k+1)-2r_{nk},...,(\alpha_{nk}-1)(k+1)-2r_{nk}\},$$
$$\dots$$
$$S_l=\{-lr_{nk},(k+1)-lr_{nk},2(k+1)-lr_{nk},...,(\alpha_{nk}-1)(k+1)-lr_{nk}\},$$
$$\dots$$
$$S_{t-1}=\{-lr_{nk},(k+1)-lr_{nk},2(k+1)-lr_{nk},...,(\alpha_{nk}-1)(k+1)-lr_{nk}\},$$
$$R=\{-(l+1)r_{nk},(k+1)-(l+1)r_{nk},...,(\alpha_{nk}-1)(k+1)-(l+1)r_{nk}\}.$$

Operations here are considered to be done by modulo n. Consider the elements of sets $S_0,...,S_{t-1}$ in the specified order and decompose each of the sets into $p+1$ parts (so that cardinality of the i-th set is x_i). Thus, we have constructed sets $P_0, P_1, ..., P_{(p+1)t-1}$. Now let's consider the elements of R in the specified order and separate from them the first r_{mp} sets with cardinalities $x_{p+1}, x_{p+2}, ..., x_{p+r_{mp}}$ correspondingly. Thus, we get sets $P_0, P_1, ..., P_{m-1}$ in C_n^k graph. Since the vector from x_i is an optimal solution, it is easy to check that

$$|S| = \alpha_{nk}\alpha_{mp} + [\alpha_{nk} r_{mp} / (p+1)] = [\rho(C_m^p)\alpha(C_n^k)].$$

To finalize the proof of the Theorem, it remains to show that the constructed set S is an independent set in the product graph. It suffices to show that any sequential $p+1$ sets in the cyclic sequence of sets $P_0, P_1, ..., P_{m-1}$ are pairwise disjoint and the union of the $p+1$ sets is an independent set in C_n^k.

Let $P_{i(\bmod n)}, P_{(i+1)(\bmod n)}, ..., P_{(i+p)(\bmod n)}$ be such a sequence of sets. If P_0 and P_{m-1} aren't present in the sequence together, then the statement follows from the construction, otherwise, taking into consideration the fact that the vector from x_i is a solution of (2), the statement follows from the definition of number l. Indeed, the last element (in the above mentioned order) of P_{m-1} is $-(l+1)r_{nk} + (k+1)([\alpha_{nk} r_{mp} / (p+1)] - 1) \le -(k+1)$, which is not adjacent to the first element (in the mentioned order) of P_0, that is 0. The Theorem is thus proved.

Corollary. For any generalized cycles C_m^p and C_n^k holds $\alpha(C_m^p \times C_n^k) =$ $= \min([\rho(C_m^p) \times \alpha(C_n^k)], [\alpha(C_m^p) \times \rho(C_n^k)])$, particularly, $\alpha(C_n^k \times C_n^k) = \left[\frac{[n/k]n}{k}\right]$.

Proof. Obviously, it's enough to prove only the first equality, which is a direct consequence of the Theorem and the fact from [5, 6]:
$\alpha(C_m^p \times C_n^k) \le \min(\rho(C_m^p) \times \alpha(C_n^k), \alpha(C_m^p) \times \rho(C_n^k))$. The corollary is thus proved.

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Ս. Հ. Բադալյան, Ս. Ե. Մարկոսյան

Ընդհանրացված ցիկլերի ուժեղ արտադրյալի անկախության թվի մասին

Սույն աշխատանքում ուսումնասիրված է ընդհանրացված ցիկլերի ուժեղ արտադրյալի անկախության թիվը: Տրված է արտադրյալ գրաֆում ամենամեծ անկախ բազմությունը կառուցելու եղանակ: Եղանակը մասնա-վորապես հիմնված է յուրահատուկ կոմբինատոր օպտիմալացման խնդրի վրա, որը նույնպես լուծված է: Ստացված հիմնական արդյունքը կենտ ցիկլերի համար ստացված նման արդյունքի ընդհանրացումն է [6].

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О числе независимости произведения обобщенных циклов

В настоящей работе исследуется число независимости произведения обобщенных циклов. Дан метод для построения наибольшего независимого множества в графе произведения. Метод, в частности, основан на специфической комбинаторной задаче, которая тоже решена. Полученный результат является обобщением известного факта для нечетных циклов [6].