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Mathematics

ON A GENERALIZED ENTROPIC PROPERTY

AMIR EHSANI*

Islamic Azad University, Mahshahr Branch, Iran

Let $\cdot, *$ be idempotent operations on the set *A*, and $*$ be a commutative operation. We prove, that if the pair $(\cdot, *)$ satisfies the generalized entropic property, then $(\cdot, *)$ is entropic.

Keywords: complex algebra, mode, entropic algebra, generalized entropic property.

Introduction. For an algebra $A=(A,F)$ we define the complex operations for every $\phi \neq A_1,...,A_n \subseteq A$ and *n*-ary $f \in F$ on the set $\rho(A)$ of all non-empty subsets of the set *A* by equality $f(A_1,...,A_n)=\{f(a_1,...,a_n): a_i \in A_i\}$. The algebra $Cm\,A=(\rho(A),F)$ is called a complex algebra of *A*.

Complex algebras were studied by several authors (G. Grätzer and H. Lakser [1], C. Brink [2], I. Bošnjak and R. Madarász [3], A. Romanowska and J. D. H. Smith [4–6], K. Adaricheva, A. Pilitowska, D. Stanovský [7] and others). The concept of complex operations is widely used. In group theory, for instance, a co-set *xN* is the complex product of the singleton $\{x\}$ and the subgroup *N*. For a lattice *L* the set *Id L* of its ideals is a lattice with regard to set inclusion. If *L* is distributive, then its union and intersection in *Id L* are the complex operations, obtained from union and intersection of *L,* and, therefore, *Id L* is a subalgebra of *Cm L.*

Now, consider the set *CSub A* of all (non-empty) subalgebras of algebra *A*. This set may or may not be closed with regard to complex operations. For instance, if *A* is an Abelian group, it is, however, non-closed for most groups. In the case above, *CSub A* is a subsystem of *Cm A*, and we call it a complex algebra of subalgebras. We say that *A* has the complex algebra of subalgebras or that *CSub A* exists.

An algebra $A=(A,F)$ is called entropic (or medial), if for any *n*-ary $f \in F$ and *m*-ary $g \in F$ it satisfies the mediality identity:

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm}))=f(g(x_{11},...,x_{1m}),...,g(x_{n1},...,x_{nm})). \qquad (1)$$

* E-mail: A.Ehsani@mahshahriau.ac.ar

So, algebra A is medial, if it satisfies the hyperidentity of mediality [8–10]. Note that a groupoid is entropic, iff it satisfies the identity of mediality $xy \cdot uv \approx xu \cdot yv$ [11].

A variety V is called entropic (or medial), if any algebra is entropic in V. Algebra A is called idempotent (commutative), if any operation of A is idempotent (commutative). An idempotent entropic algebra is called a mode [6].

The Generalized Entropic Property.

Definition 1. We say that a variety V (the algebra A) satisfies the generalized entropic property, if for any n-ary operation f and m-ary operation g of V (of A) there exist m-ary terms $t_1,...,t_n$, such that the identity

$$g(f(x_{11},...,x_{n1}),...,f(x_{1m},...,x_{nm})) = f(t_1(x_{11},...,x_{1m}),...,t_n(x_{n1},...,x_{nm})) \qquad (2)$$

holds in V (in A).

For example, a groupoid satisfies the generalized entropic property, if there are binary terms t and s, such that the identity $xy \cdot uy \approx t(x,y) \cdot s(y,v)$ holds.

It was proved by T. Evans [12], that for the variety V of groupoids any groupoid in V has a complex algebra of subalgebras, iff V satisfies the above identity for some t and s.

Theorem 1. Every algebra in a variety V has a complex algebra of subalgebras, iff the variety V satisfies the generalized entropic property.

The proof is given in [7].

The Main Results. The generalized entropic property for algebra $A = (A, \cdot, *)$ with two binary operations means that the following identities hold:

$$(x \cdot y) \cdot (u \cdot v) \approx t_1(x,u) \cdot s_1(y,v),$$
$$(x * y) * (u * v) \approx t_2(x,u) * s_2(y,v),$$
$$(x * y) \cdot (u * v) \approx t_3(x,u) * s_3(y,v),$$
$$(x \cdot y) * (u \cdot v) \approx t_4(x,u) \cdot s_4(y,v).$$

Immediate consequences of the generalized entropic property in idempotent algebra $A = (A, \cdot, *)$ with two binary operations are the following identities that can be treated as laws of pseudo-distributivity:

$$(x * y) \cdot (x * z) \approx x * s(y,z), \quad (y * x) \cdot (z * x) \approx t(y,z) * x.$$

Entropic law for the algebra $A = (A, \cdot, *)$ implies the following identities:

$$(x \cdot y) \cdot (u \cdot v) \approx (x \cdot u) \cdot (y \cdot v), \qquad \text{(a)}$$
$$(x * y) * (u * v) \approx (x * u) * (y * v), \qquad \text{(b)}$$
$$(x * y) \cdot (u * v) \approx (x \cdot u) * (y \cdot v). \qquad \text{(c)}$$

Definition 2. Let g and f be m-ary and n-ary operations on the set A. We say, that pair of the operations *(g, f)* satisfies the generalized entropic property, if there exist terms $t_1,...,t_n$ of the algebra $A = (A, f, g)$, such that identity (2) holds in the algebra $A = (A, f, g)$. The *(g, f)* operation pair is called entropic (or medial), if identity (1) holds in the algebra $A = (A, f, g)$.

Theorem 2. Let $A = (A, \cdot, *)$ be an idempotent algebra. If $*$ is commutative and the pair $(\cdot, *)$ satisfies the generalized entropic property, then $(\cdot, *)$ is entropic.

Proof. To prove (c), from the generalized entropic property we obtain:

$$(x*y)\cdot(u*v)\approx t(x,u)*s(y,v).$$

Using the following pseudo-distributive properties

$$(x*y)\cdot(x*z)\approx x*s(y,z),\quad (y*x)\cdot(z*x)\approx t(y,z)*x$$

and commutativity of $*$, we get

$$t(x,u)*z\approx(x*z)\cdot(u*z)\approx(z*x)\cdot(z*u)\approx z*s(x,u)\approx s(x,u)*z.$$

So, using idempotency and commutativity of $*$, we have

$$t(x,u)\approx t(x,u)*t(x,u)\approx s(x,u)*t(x,u)\approx t(x,u)*s(x,u)\approx(x*x)\cdot(u*u)\approx x\cdot u.$$

In the same manner we obtain $s(y,v)\approx y\cdot v$. Thus, from the generalized entropic property and the last two identities we have

$$(x*y)\cdot(u*v)\approx t(x,u)*s(y,v)\approx(x*u)\cdot(y*v).$$

Corollary 1. Let $A=(A,\cdot,*)$ be an idempotent and commutative algebra with two binary operations. If A satisfies the generalized entropic property, then A is an entropic groupoid.

Corollary 2. Every idempotent and commutative algebra with binary operations, satisfying the generalized entropic property, is an entropic groupoid.

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Ամիր Էհսանի

Ընդհանրացված էնտրոպիկության հատկության մասին

Այս հոդվածում ապացուցվում է, որ եթե միևնույն բազմության վրա որոշված երկու իդեմպոտենտ գործողություններ (որոնցից մեկը կոմուտատիվ է) բավարարում են ընդհանրացված էնտրոպիկության հատկությանը, ապա դրանք կբավարարեն նաև էնտրոպիկության հատկությանը:

Амир Эхсани.

Об обобщенном энтропичном свойстве

Пусть $\cdot,*$ суть идемпотентные операции на множестве A, и $*$ – коммутативная операция. Мы доказываем, что если пара $(\cdot,*)$ удовлетворяет обобщенному энтропичному свойству, то пара $(\cdot,*)$ энтропична.