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Mathematics

ON SOME SINGULAR INTEGRAL EQUATIONS ON THE SEMI-AXIS

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In this paper the non-characteristic singular integral equations on the semi-axis are discussed. The solution of these equations is reduced to the solution of one-dimensional pseudo-differential equations. Some examples of singular equations for which explicit solutions exist are provided.

***Keywords*:** singular equation, pseudo-differential equation, matrix coupling.

1⁰. Let $\overline{\mathbb{R}}$ be a two-point compactification of $\mathbb{R}=(-\infty,+\infty)$ and $C\left(\overline{\mathbb{R}}\right)$ – the Banach algebra of all complex valued continuous function on $\overline{\mathbb{R}}$. What follows, $\rho(z)=z^{\beta}$ $(-1<\beta<1)$ is understood as the branch of this function that is analytical in $\mathbb{C}\setminus(-\infty,0)$ and assumes positive values on the positive semi-axis $\mathbb{R}_+=(0,+\infty)$. Let Σ_β be a subalgebra of all linear bounded operators acting from and to the weighted space $L_2(\mathbb{R}_+,\rho)$, generated by operators I and $S_{\mathbb{R}_+}$, where $S_{\mathbb{R}_+}$ is a singular integral operator acting along $\mathbb{R}_+$:

$$\left(S_{\mathbb{R}_+}y\right)(x)=\frac{1}{\pi i}\int_0^{+\infty}\frac{y(\tau)}{\tau-x}d\tau .$$

Here the integral is perceived in terms of the main value. The notation F means the following Fourier transform:

$$(Fy)(\xi)=\hat{y}(\xi)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ix\xi}y(x)dx \quad \left(y\in L_2(\mathbb{R}),\xi\in\mathbb{R}\right),$$

and γ is the operator continuously reflecting $L_2(\mathbb{R}_+,\rho)$ in $L_2(\mathbb{R})$ according to formula $(\gamma y)(x)=e^{sx}y(e^{\alpha x})$, where $\alpha>0$, $\sigma=\alpha(\beta+1)/2$ and $s=\sigma+i\zeta$. It is not difficult to ascertain that the operator $\gamma^{-1}F$ is the Mellin transform.

Below the multiplicative operator by function u is denoted as Λ_u (i.e. $\Lambda_u y=uy$). For any function $a\in C\left(\overline{\mathbb{R}}\right)$ the operator $K_a=\gamma^{-1}F\Lambda_aF^{-1}\gamma$ belongs to algebra Σ_β (see, e.g., [1, 2]).

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As is well known, the characteristic equation $\left(\Lambda_a + S_{\mathbb{R}_+}\Lambda_b\right)y = f$ $\left(y, f \in L_2\left(\mathbb{R}_+, \rho\right)\right)$ admits the explicit solution [3]. The present paper is devoted to the investigation of "difficult" singular integral operators of a type $\widetilde{K} = \sum\limits_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m}$ $\left(\varphi_m \in C\left(\overline{\mathbb{R}}\right), \psi_m \in L_\infty\left(\mathbb{R}_+\right)\right)$, acting from $L_2\left(\mathbb{R}_+, \rho\right)$ to itself.

Let $\mathbb{H}_r\left(\mathbb{R}\right)$ $\left(r \geq 0\right)$ be Sobolev–Slobodetski spaces of the generalized functions u, the Fourier transform $\hat{u}$ of which belongs to $L_2\left(\mathbb{R}, \left(1+|x|\right)^r\right)$ space. Following [4], the class of locally integrable functions A on $\mathbb{R}$, complying to condition $\left|A\left(\xi\right)\right| \leq c\left(1+\left|\xi\right|\right)^r$, is denoted as $\mathbb{S}_r^0$.

Let $\psi_0\left(x\right) = \left(1 + \left|\alpha^{-1}\ln x\right|\right)^r$ $\left(x \in \mathbb{R}_+\right)$ and functions $\psi_i \in L_\infty\left(\mathbb{R}_+\right)$ $\left(i = 1, \ldots, N\right)$. Now determine the functions $A_0 = \psi_0 \circ l,$ $A_m = \left(\psi_m \circ l\right)A_0$ $\left(m = 1, \ldots, N\right)$ on $\mathbb{R}$, where $l\left(x\right) = e^{\alpha x}$ $\left(x \in \mathbb{R}\right)$. It follows from here that $\psi_0 = A_0 \circ h$ and $\psi_m = \left(A_m \circ h\right)\psi_0^{-1}$ $\left(m = 1, \ldots, N\right)$, where $h\left(x\right) = \alpha^{-1}\ln x$ when $x \in \mathbb{R}_+$. It is obvious that $A_m \in \mathbb{S}_r^0$ for all $m = 0, \ldots, N$.

Let us consider a operator $A\left(x, D\right)$ with a symbol $A\left(x, t\right) = \sum\limits_{m=1}^{N}\varphi_m\left(x\right)A_m\left(t\right)$:

$$A\left(x, D\right)u = \int\limits_{-\infty}^{\infty} e^{-ixt}A\left(x, t\right)\hat{u}\left(t\right)dt.$$

Since $A\left(x, D\right)u = \sum\limits_{m=1}^{N}\Lambda_{\varphi_m}A_m\left(D\right)u$, then for $r \geq 0$ the reflection $A\left(x, D\right)$ may be prolonged till the continuous reflection from $\mathbb{H}_r\left(\mathbb{R}\right)$ to $L_2\left(\mathbb{R}\right)$(see [4]).

Below the solution of "difficult" equation $\widetilde{K}z = g$ is reduced to a pseudo-differential equation $A\left(x, D\right)y = f$ for some f. The examples of "difficult" singular integral equations investigated based on this relationship are provided.

2⁰. Let $X_i\left(i = 1, \ldots, 4\right)$ be linear spaces on the field of complex numbers, $\omega_1 : X_1 \to X_2,$ $\omega_2 : X_2 \to X_1,$ $\omega_3 : X_1 \to X_3,$ $\omega_4 : X_4 \to X_1$ are linear reflections and ω_4 is an invertible reflection. Let us consider the linear reflections

$$T : X_2 \to X_2 \oplus X_3, \qquad K : X_4 \to X_1 \oplus X_3,$$
$$A_{12} : X_1 \oplus X_3 \to X_2 \oplus X_3, \quad A_{21} : X_2 \to X_4,$$
$$A_{22} : X_1 \oplus X_3 \to X_4, \qquad B_{11} : X_2 \oplus X_3 \to X_2,$$
$$B_{12} : X_4 \to X_2, \qquad B_{21} : X_2 \oplus X_3 \to X_1 \oplus X_3,$$

determined by equations

$$T = \begin{bmatrix} I_{X_2} + \omega_1\omega_2 \\ \omega_3\omega_2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -\omega_1 & 0 \\ -\omega_3 & I_{X_3} \end{bmatrix},$$

$$A_{21}=\left[-\omega_4^{-1}\omega_2\right],\quad A_{22}=\left[\omega_4^{-1}\quad 0\right],\quad B_{11}=\left[I_{X_2}\quad 0\right],\quad B_{12}=\left[\omega_1\omega_4\right],$$

$$B_{21}=\begin{bmatrix}\omega_2 & 0\\ 0 & I_{X_3}\end{bmatrix},\quad K=\begin{bmatrix}\omega_4+\omega_2\omega_1\omega_4\\ \omega_3\omega_4\end{bmatrix}.$$

By means of direct calculations it is easy to make sure of the validity of equality

$$\begin{bmatrix}T & A_{12}\\ A_{21} & A_{22}\end{bmatrix}^{-1}=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & K\end{bmatrix}.$$

Based on this equality and results of [5] the following statement is derived.

Assignment 1. If $K^{(-1)}$ is pseudo-inverse of K, then the reflection

$$T^{(-1)}=B_{11}-B_{12}K^{(-1)}B_{21}$$

is pseudo-inverse of T. Similarly, if $T^{(-1)}$ is pseudo-inverse of T, then

$$K^{(-1)}=A_{22}-A_{21}T^{(-1)}A_{12}$$

is pseudo-inverse of K. Spaces $\mathrm{Ker}T$ and $\mathrm{Ker}K$ are isomorphic. Besides, the following equalities are valid:

$$\mathrm{Ker}T=B_{12}\mathrm{Ker}K,\quad \mathrm{Ker}K=A_{21}\mathrm{Ker}T,$$

$$\mathrm{Im}\,T=B_{21}^{-1}\,\mathrm{Im}\,K,\quad \mathrm{Im}\,K=A_{12}^{-1}\,\mathrm{Im}\,T.$$

3⁰. Let $\mathbb{M}_r(\mathbb{R})$ be some direct addition to the linear space $\mathbb{H}_r(\mathbb{R})$ in $L_2(\mathbb{R})$ space, and

$$\pi_1:L_2(\mathbb{R})\to\mathbb{H}_r(\mathbb{R}),\ \pi_2:L_2(\mathbb{R})\to\mathbb{M}_r(\mathbb{R})$$

are projection operators that are related by the ratio

$$\pi_1y+\pi_2y=y,\ \ y\in L_2(\mathbb{R}).$$

Let us determine a space $W=\left\{f:\psi_0^{-1}f\in L_2(\mathbb{R}_+,\rho)\right\}$. Then, apply the Assignment 1 for spaces

$$X_1=L_2(\mathbb{R}_+,\rho),\ X_2=\mathbb{H}_r(\mathbb{R}),\ X_3=\mathbb{M}_r(\mathbb{R}),\ X_4=W$$

and operators

$$\omega_1=\pi_1F^{-1}\gamma,\ \omega_2=\sum_{m=1}^{N}K_{\varphi_m}\gamma^{-1}FA_m(D)-\gamma^{-1}F,\ \omega_3=\pi_2F^{-1}\gamma,\ \omega_4=\Lambda_{\psi_0^{-1}}.$$

By using the equality $F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F=\Lambda_{\varphi_m}$ $(m=1,\dots,N)$ it is easy to ascertain that the operator T acting from $\mathbb{H}_r(\mathbb{R})$ to $L_2(\mathbb{R})$ coincides with $A(x,D)$.

Lemma 1. Let

$$f=(\tilde f,0)\in L_2(\mathbb{R}_+,\rho)\oplus\mathbb{M}_r(\mathbb{R}).$$

The function z is the solution of equation $Kz=f$, iff $z\in L_2(\mathbb{R}_+,\rho)$ and $\tilde Kz=\tilde f$.

Proof. Let z be the solution of equation $Kz=f$, that is

$$\omega_4z+\omega_2\omega_1\omega_4z=\tilde f,\quad \omega_3\omega_4z=0. \tag{1}$$

First prove that $z\in L_2(\mathbb{R}_+,\rho)$. Really, from the second equation of (1) it follows that $g=F^{-1}\gamma\Lambda_{\psi_0^{-1}}z\in\mathbb{H}_r(\mathbb{R})$. Consequently, $\gamma z=FA_0(D)g\in L_2(\mathbb{R})$, that is $z \in L_2(\mathbb{R}_+,\rho)$. Note now that from (1) it also follows that $\pi_1 F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$. From the first equation of (1) it follows that $\tilde{K}z = \tilde{f}$.

Let $z \in L_2(\mathbb{R}_+,\rho)$ and $\tilde{K}z = \tilde{f}$. From equality $F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = A_0^{-1}(D)F^{-1}\gamma z$ it follows that

$$\omega_3\omega_4 z = 0 .$$

Since $\tilde{K}z = \tilde{f}$, we have

$$\left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m}\right)z = \left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{A_m\circ h}\Lambda_{\psi_0^{-1}}\right)z = \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{A_m\circ h} - I\right)\Lambda_{\psi_0^{-1}}\right)z =$$

$$= \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D) - \gamma^{-1}F\right)F^{-1}\gamma\Lambda_{\psi_0^{-1}}\right)z =$$

$$= \left(\Lambda_{\psi_0^{-1}} + \left(\sum_{m=1}^{N} K_{\varphi_m}\gamma^{-1}FA_m(D) - \gamma^{-1}F\right)\pi_1 F^{-1}\gamma\Lambda_{\psi_0^{-1}}\right)z =$$

$$= (\omega_4 + \omega_2\omega_1\omega_4)z = \tilde{f}.$$

Thus, the equalities are correct, that is $Kz = f$. The proof is completed.

Theorem 1. To ensure that function $z \in L_2(\mathbb{R}_+,\rho)$ satisfies the equation

$$\sum_{m=1}^{N} K_{\varphi_m}\Lambda_{\psi_m} z = g\,(g \in L_2(\mathbb{R}_+,\rho)), \qquad (2)$$

it is necessary and sufficient that function $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$ satisfy the equation

$$A(x,D)y = F^{-1}\gamma g\,(F^{-1}\gamma g \in L_2(\mathbb{R})). \qquad (3)$$

Proof. Let the function $z \in L_2(\mathbb{R}_+,\rho)$ be a solution of equation (2). Acting by operator $F^{-1}\gamma$ on both the parts of the equation (2) (it is feasible, since the functions ψ_m $(m = 1,\ldots,N)$ are bounded), we obtain

$$\sum_{m=1}^{N} F^{-1}\gamma K_{\varphi_m}\Lambda_{A_m\circ h}\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma g .$$

Taking into account the equalities $\Lambda_{A_m\circ h} = \gamma^{-1}FA_m(D)F^{-1}\gamma$ and $F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F = \Lambda_{\varphi_m}$ $(m = 1,\ldots,N)$, we find

$$\sum_{m=1}^{N} \Lambda_{\varphi_m} A_m(D)y = F^{-1}\gamma g .$$

Earlier, it has been shown that $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z \in \mathbb{H}_r(\mathbb{R})$ (see the proof of Lemma 1). Vice versa, now let the function $y \in \mathbb{H}_r(\mathbb{R})$ be the solution of equation (3). Then it is true that $z = \Lambda_{\psi_0}\gamma^{-1}Fy \in L_2(\mathbb{R}_+,\rho)$, the proof of which is provided in Lemma 1. At the insertion of $y = F^{-1}\gamma\Lambda_{\psi_0^{-1}} z$ into (3), we have

$$\sum_{m=1}^{N} \Lambda_{\varphi_m} A_m(D)F^{-1}\gamma\Lambda_{\psi_0^{-1}} z = F^{-1}\gamma g .$$

Taking into account the equality $\Lambda_{\varphi_m}=F^{-1}\gamma K_{\varphi_m}\gamma^{-1}F \quad (m=1,\dots,N)$, we obtain

$$\sum_{m=1}^{N}F^{-1}\gamma K_{\varphi_m}\gamma^{-1}FA_m(D)F^{-1}\gamma\Lambda_{\psi_0^{-1}}z=F^{-1}\gamma g\,. \tag{4}$$

Acting by operator $\gamma^{-1}F$ on both the parts of equality (4), we find

$$\sum_{m=1}^{N}K_{\varphi_m}\gamma^{-1}FA_m(D)F^{-1}\gamma\Lambda_{\psi_0^{-1}}z=g\,.$$

By using equality $\gamma^{-1}FA_m(D)F^{-1}\gamma=\Lambda_{A_m\circ h} \quad (m=1,\dots,N)$, we find that $\sum_{m=1}^{N}K_{\varphi_m}\Lambda_{\psi_m}z=g$, i.e. the function z is the solution of equation (2).

The Theorem is proved.

4⁰. In case of $N=2$, $\varphi_1(x)=1$ and $\varphi_2(x)=\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}$ we obtain $K_{\varphi_1}=I$ and $(K_{\varphi_2}y)(x)=\frac{1}{\pi^2}\int_0^{+\infty}y(\xi)\frac{\ln\xi-\ln x}{\xi-x}d\xi$.

Now, let us highlight two examples.

Example 1. Let $A_1(\xi)=-\xi^2-\frac{\pi^2}{\alpha^2}$, $A_2(\xi)=2\frac{\pi^2}{\alpha^2}$ and $r=2$, then we have

$$A(x,D)y(x)=y''(x)+\frac{\pi^2}{\alpha^2}\left(2\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}-1\right)y(x),$$

$$\widetilde{K}z(x)=\frac{-\left(\alpha^{-1}\ln x\right)^2-\pi^2/\alpha^2}{\left(1+\left|\alpha^{-1}\ln x\right|\right)^2}z(x)+\frac{2}{\alpha^2}\int_0^{+\infty}\frac{z(\xi)}{\left(1+\left|\alpha^{-1}\ln\xi\right|\right)^2}\cdot\frac{\ln\xi-\ln x}{\xi-x}d\xi\,.$$

Taking into account that functions

$$y_1(x)=e^{\frac{\pi}{\alpha}x}\left[1+e^{2\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)}\right]^{-1},\quad y_2(x)=e^{-\frac{\pi}{\alpha}x}\left[1+e^{-2\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)}\right]^{-1}$$

are solutions of equation $A(x,D)y=0$ from class $\mathbb{H}_2(\mathbb{R})$, we obtain according to the Assignment 1 that functions $z_k(x)=-\omega_4^{-1}\omega_2 y_k(x)\ \ (k=1,2)$ are the basis of $\operatorname{Ker}\widetilde{K}$.

Example 2. Let $A_1(\xi)=-a_1i\xi+a_2$, $A_2(\xi)=b_1i\xi-b_2$ and $r=1$, where $a_1,a_2,b_1,b_2\in\mathbb{C}$, then we find

$$A(x,D)y(x)=\left(a_1-b_1\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}\right)y'(x)+\left(a_2-b_2\left[\operatorname{ch}\frac{\pi}{\alpha}\left(x-\xi+i\frac{\alpha\beta}{2}\right)\right]^{-2}\right)y(x),$$

$$\widetilde{K}z(x)=\frac{-a_1i\alpha^{-1}\ln x+a_2}{1+\left|\alpha^{-1}\ln x\right|}z(x)-\frac{1}{\pi^2}\int_0^{+\infty}\frac{-b_1i\alpha^{-1}\ln\xi+b_2}{1+\left|\alpha^{-1}\ln\xi\right|}\cdot\frac{\ln\xi-\ln x}{\xi-x}z(\xi)d\xi\,.$$

Taking into account that the only solution of equation $A(x,D)y(x)=0$ does not belong to class $L_2(\mathbb{R})$, we obtain that equation $\widetilde{K}z(x)=0$ does not have any solutions from $L_2(\mathbb{R}_+,\rho)$.

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