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Mathematics

ON BOUNDED OPERATORS IN L^p SPACES

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In the present paper the linear operators that depend on a normal pair of weight functions $\{\varphi,\psi\}$ in the Banach spaces $L^p(D)$, where D is the unit disk in the complex plane, are considered. It is investigated, for which values of p these operators are bounded.

***Keywords*:** Banach space, analytic function, bounded operator.

Introduction. The paper is devoted to investigation of some linear operators depending on a normal pair of weight functions in $L^p(D)$ spaces. The concept of a normal pair of weight functions introduced for the first time by Shields and Williams [1] and was appeared convenient for statement of the estimates of integrals and for exposition of projectors in weight spaces. The basic result of paper is formulated in Theorem 2. The case $p=1$ is discussed in [1]. Note that the case of power weight functions is considered in [2] (Chapter 1, § 2, Theorem 1.9).

We use the following notations: $D=\{z\in\mathbb{C}:\ |z|<1\}$ is the unit disk in the complex plane $\mathbb{C}$, dA is the normalized area measure on D. In terms of real coordinates we have $dA(z)=\frac{1}{\pi}dx\,dy=\frac{1}{\pi}rdr\,d\theta,\quad z=x+iy=re^{i\theta}$.

We define $L^p(D,dA)$ as the set of all functions f that are measurable by dA in D, for which

$$\|f\|_p=\left\{\int_D|f(w)|^p\,|\,dA(w)|\right\}^{1/p}<+\infty,\quad 0<p<\infty.$$

Let the functions $\varphi(r)$ and $\psi(r)$ be positive and continuous on $(0,1]$ with

$$\lim_{r\to1}\varphi(r)=0\quad\text{and}\quad\int_0^1\psi(r)\,dr<\infty.$$

Definition 1. A function $\varphi(r)$ is called normal, if there exist $k>\varepsilon>0$ and $r_0<1$ such that

$$\frac{\varphi(r)}{(1-r)^{\varepsilon}}\searrow 0,\qquad \frac{\varphi(r)}{(1-r)^{k}}\nearrow\infty\qquad (r_0\le r,\ \ r\to1-0). \tag{1}$$

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Definition 2. The functions $\{\varphi,\psi\}$ is called a normal pair, if φ *is* normal and if for some k satisfying (1), there exists $\alpha > k-1$ such that

$$\varphi(r)\psi(r) = (1-r^2)^{\alpha}, \quad 0 \le r < 1.$$

Consider the integral operator

$$Tf(z) = \int_D \frac{\psi(z)\varphi(w)}{\left|1-z\overline{w}\right|^{2+\alpha}} f(w)\,dA(w). \tag{2}$$

We want to establish for which p this operator is bounded.

Auxiliary Lemmas.

Lemma 1. For $\gamma > -1$ and $m > 1+\gamma$ we have

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\gamma}\,dr \le C(1-\rho)^{1+\gamma-m}, \quad 0 < \rho < 1,$$

where the constant $C = C(m,\gamma)$ does not depend on ρ. For the proof see [1] (Chapter 1, § 3).

Lemma 2. For a positive $\beta > 0$ and $z \in D$

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+\beta}} = O\left(\frac{1}{(1-|z|^2)^{\beta}}\right).$$

For the proof see [3] (Chapter 4, § 6).

Lemma 3. For $k-\alpha < \dfrac{\beta}{p} < 1+\varepsilon$ there exist a constant M_0 such that

$$\int_D \frac{\varphi(w)}{(1-|w|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}} dA(w) \le M_0 \frac{1}{\psi(z)(1-|z|^2)^{\beta/p}}.$$

Proof. According to the definition of a normal pair of functions, $\varphi(r)\psi(r) = (1-r^2)^{\alpha}$ for $0 \le r < 1$, and it will be sufficient to show that

$$\int_D \frac{\varphi(w)}{(1-|w|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}} dA(w) \le M_0 \frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}}.$$

Let $w = re^{i\theta}$. As $\alpha+1>0$, then, due to Lemma 2, there exists a constant C such that $\displaystyle\int_0^{2\pi} \frac{1}{|1-zre^{-i\theta}|^{\alpha+2}} d\theta \le \frac{C}{(1-(|z|r)^2)^{\alpha+1}}$.

Passing to polar coordinates, we receive

$$\begin{aligned} &\int_D \frac{\varphi(w)}{(1-|w|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}} dA(w) = \frac{1}{\pi}\int_0^1 \frac{r\varphi(r)}{(1-r^2)^{\beta/p}} \int_0^{2\pi} \frac{d\theta}{|1-zre^{-i\theta}|^{\alpha+2}} dr \le \\ &\le C\int_0^1 \frac{\varphi(r)}{(1-|r|^2)^{\beta/p}(1-(|z|r)^2)^{\alpha+1}} dr \le C_1\int_0^1 \frac{\varphi(r)}{(1-|r|)^{\beta/p}(1-|z|r)^{\alpha+1}} dr. \end{aligned} \tag{3}$$

Divide the last integral into three parts:

$$\begin{aligned} &\int_0^1 \frac{\varphi(r)}{(1-r)^{\beta/p}(1-|z|r)^{\alpha+1}} = \int_0^{r_0} \frac{\varphi(r)}{(1-r)^{\beta/p}(1-|z|r)^{\alpha+1}} dr + \\ &+\int_{r_0}^{|z|} \frac{\varphi(r)}{(1-r)^{\beta/p}(1-|z|r)^{\alpha+1}} dr + \int_{|z|}^1 \frac{\varphi(r)}{(1-r)^{\beta/p}(1-|z|r)^{\alpha+1}} dr = I_1+I_2+I_3. \end{aligned} \tag{4}$$

Obviously I_1 is bounded for all z. Therefore, there exists a constant C_2 such that

$$I_1 \le C_2 \frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}}. \tag{5}$$

From the definition of normal function (1) we have $\frac{\varphi(r)}{(1-r)^k} \le \frac{\varphi(z)}{(1-|z|)^k}$ for $r_0 < r \le |z|$. Therefore,

$$I_2 = \int_{r_0}^{|z|} \frac{\varphi(r)}{(1-r)^{\beta/p}(1-|z|r)^{\alpha+1}} dr = \int_{r_0}^{|z|} \frac{\varphi(r)(1-r)^{k-(\beta/p)}}{(1-r)^k (1-|z|r)^{\alpha+1}} dr \le \frac{\varphi(z)}{(1-|z|)^k} \int_{r_0}^{|z|} \frac{(1-r)^{k-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr.$$

As $k-\alpha < \frac{\beta}{p} < 1+\varepsilon$, then $k-\frac{\beta}{p} < \alpha$ and $-1 < \varepsilon - \frac{\beta}{p} < k - \frac{\beta}{p}$, that is $-1 < k - \frac{\beta}{p} < \alpha$, and taking into account Lemma 1 we have

$$\int_{r_0}^{|z|} \frac{(1-r)^{k-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le \int_0^1 \frac{(1-r)^{k-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le C_3 \frac{1}{(1-|z|)^{(\beta/p)+\alpha-k}}$$

for some constant C_3. Therefore,

$$I_2 \le \frac{\varphi(z)}{(1-|z|)^k} \int_{r_0}^{|z|} \frac{(1-r)^{k-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le \frac{\varphi(z)}{(1-|z|)^k} \frac{C_3}{(1-|z|)^{(\beta/p)+\alpha-k}} \le C_4 \frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}}. \tag{6}$$

Now we shall pass to I_3. As $\frac{\varphi(r)}{(1-r)^\varepsilon} \le \frac{\varphi(z)}{(1-|z|)^\varepsilon}$ for $|z| < r < 1$, then

$$I_3 = \int_{|z|}^1 \frac{\varphi(r)}{(1-r)^{(\beta/p)}(1-|z|r)^{\alpha+1}} dr = \int_{|z|}^1 \frac{\varphi(r)(1-r)^{\varepsilon-(\beta/p)}}{(1-r)^\varepsilon (1-|z|r)^{\alpha+1}} dr \le$$

$$\le \frac{\varphi(z)}{(1-|z|)^\varepsilon} \int_{|z|}^1 \frac{(1-r)^{\varepsilon-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le \frac{\varphi(z)}{(1-|z|)^\varepsilon} \int_0^1 \frac{(1-r)^{\varepsilon-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr.$$

We have also $k-\alpha < \frac{\beta}{p} < 1+\varepsilon$, therefore, $\varepsilon - \frac{\beta}{p} > -1$ and $\varepsilon - \alpha < k - \alpha < \frac{\beta}{p}$, i.e. $\alpha + 1 > 1 + \varepsilon - \frac{\beta}{p}$. Again using Lemma 1 we get

$$\int_0^1 \frac{(1-r)^{\varepsilon-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le C_5 \frac{1}{(1-|z|)^{(\beta/p)+\alpha-\varepsilon}},$$

whence we have

$$I_3 \le \frac{\varphi(z)}{(1-|z|)^\varepsilon} \int_0^1 \frac{(1-r)^{\varepsilon-(\beta/p)}}{(1-|z|r)^{\alpha+1}} dr \le C_5 \frac{\varphi(z)}{(1-|z|)^{(\beta/p)+\alpha}} \le C_6 \frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}} \tag{7}$$

with some constants C_5 and C_6. Combining the obtained results (3)–(7), we get the constant M_0 such that

$$\int_D \frac{\varphi(w)}{(1-|w|^2)^{(\beta/p)} \left|1-z\overline{w}\right|^{2+\alpha}} dA(w) \le M_0 \frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}} = M_0 \frac{1}{\psi(z)(1-|z|^2)^{\beta/p}}$$

for $k-\alpha<\dfrac{\beta}{p}<1+\varepsilon$.

The Lemma is proved.

Lemma 4. Let $-\varepsilon<\dfrac{\beta}{q}<\alpha+1-k$. There exists a constant M_1 such that

$$\int_D \frac{\psi(z)}{(1-|z|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}}dA(z)\le M_1\frac{1}{\varphi(w)(1-|w|^2)^{\beta/p}}.$$

Proof. For $0\le r<1$ we have $\varphi(r)\psi(r)=(1-r^2)^{\alpha}$. Therefore, it is necessary to prove that

$$\int_D \frac{(1-|z|^2)^{\alpha-(\beta/p)}}{\varphi(z)\left|1-z\overline{w}\right|^{2+\alpha}}dA(z)\le M_1\frac{1}{\varphi(w)(1-|w|^2)^{\beta/p}}$$

for any constant M_1. Let $z=re^{i\theta}$. Then

$$\int_D \frac{(1-|z|^2)^{\alpha-(\beta/p)}}{\varphi(z)\left|1-z\overline{w}\right|^{2+\alpha}}dA(z)=\frac{1}{\pi}\int_0^1\frac{r(1-r^2)^{\alpha-(\beta/p)}}{\varphi(r)}\int_0^{2\pi}\frac{d\theta}{|1-\overline{w}re^{i\theta}|^{\alpha+2}}dr,$$

and, due to Lemma 2, there exists a constant K_1 such that

$$\frac{1}{\pi}\int_0^1\frac{r(1-r^2)^{\alpha-(\beta/p)}}{\varphi(r)}\int_0^{2\pi}\frac{d\theta}{|1-\overline{w}re^{i\theta}|^{\alpha+2}}dr=\frac{1}{\pi}\int_0^1\frac{r(1-r^2)^{\alpha-(\beta/p)}}{\varphi(r)}\int_0^{2\pi}\frac{d\theta}{|1-wre^{-i\theta}|^{\alpha+2}}dr\le$$

$$\le K_1\int_0^1\frac{(1-r^2)^{\alpha-(\beta/p)}}{\varphi(r)(1-(|w|r)^2)^{\alpha+1}}dr\le K_2\int_0^1\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr$$

(we have replaced $(1-r^2)$ by $(1-r)$ and $(1-(|w|r)^2)$ by $(1-|w|r)$, therefore, the value of K_1 was replaced by some other value of K_2). Dividing the last integral into three parts, we obtain

$$\int_0^1\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr=\int_0^{r_0}\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr+$$

$$+\int_{r_0}^{|w|}\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr+\int_{|w|}^{1}\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr=J_1+J_2+J_3.$$

Integral J_1 is bounded for all w, therefore, there is K_3 such that

$$J_1\le\frac{K_3}{\varphi(w)(1-|w|^2)^{\beta/p}}. \tag{8}$$

From the definition of normal function (1) we have $\dfrac{(1-r)^{\varepsilon}}{\varphi(r)}\le\dfrac{(1-|w|)^{\varepsilon}}{\varphi(w)}$, if $r_0<r\le|w|$. Therefore,

$$J_2=\int_{r_0}^{|w|}\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr=\int_{r}^{|w|}\frac{(1-r)^{\varepsilon}(1-r)^{\alpha-(\beta/p)-\varepsilon}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr\le$$

$$\le\frac{(1-|w|)^{\varepsilon}}{\varphi(w)}\int_{r_0}^{|w|}\frac{(1-r)^{\alpha-(\beta/p)-\varepsilon}}{(1-|w|r)^{\alpha+1}}dr\le\frac{(1-|w|)^{\varepsilon}}{\varphi(w)}\int_0^1\frac{(1-r)^{\alpha-(\beta/p)-\varepsilon}}{(1-|w|r)^{\alpha+1}}dr.$$

As $-\varepsilon<\dfrac{\beta}{q}<\alpha+1-k$, then $\alpha-\dfrac{\beta}{q}-\varepsilon>\alpha-\dfrac{\beta}{q}-k>-1$ and $\alpha+1>\alpha-\dfrac{\beta}{q}-\varepsilon+1$.

Using Lemma 1 we obtain $\displaystyle\int_0^1\frac{(1-r)^{\alpha-(\beta/p)-\varepsilon}}{(1-|w|r)^{\alpha+1}}dr\le K_4\frac{1}{(1-|w|)^{\varepsilon+(\beta/p)}}$ and, therefore,

$$J_2\le\frac{(1-|w|)^{\varepsilon}}{\varphi(w)}\cdot\frac{K_4}{(1-|w|)^{\varepsilon+(\beta/p)}}=\frac{K_4}{\varphi(w)(1-|w|)^{\beta/p}}\le\frac{K_5}{\varphi(w)(1-|w|^2)^{\beta/p}}. \tag{9}$$

For estimation of J_3 we use the inequality $\dfrac{(1-r)^k}{\varphi(r)}\le\dfrac{(1-|w|)^k}{\varphi(w)}$, if $|w|<r<1$.

$$J_3=\int_{|w|}^1\frac{(1-r)^{\alpha-(\beta/p)}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr=\int_{|w|}^1\frac{(1-r)^k(1-r)^{\alpha-(\beta/p)-k}}{\varphi(r)(1-|w|r)^{\alpha+1}}dr\le$$

$$\le\frac{(1-|w|)^k}{\varphi(w)}\int_{|w|}^1\frac{(1-r)^{\alpha-(\beta/p)-k}}{(1-|w|r)^{\alpha+1}}dr\le\frac{(1-|w|)^k}{\varphi(w)}\int_0^1\frac{(1-r)^{\alpha-(\beta/p)-k}}{(1-|w|r)^{\alpha+1}}dr.$$

From $-\varepsilon<\dfrac{\beta}{q}<\alpha+1-k$ we have $\alpha-k-\dfrac{\beta}{q}>-1$ and $\alpha-k-\dfrac{\beta}{q}+1<\alpha-\varepsilon-\dfrac{\beta}{q}+1<\alpha+1$.

Taking into account Lemma 1 we get $\displaystyle\int_0^1\frac{(1-r)^{\alpha-(\beta/p)-k}}{(1-|w|r)^{\alpha+1}}dr\le K_6\frac{1}{(1-|w|)^{k+(\beta/p)}}$ for a constant K_6. Therefore,

$$J_3\le\frac{(1-|w|)^k}{\varphi(w)}\cdot\frac{K_6}{(1-|w|)^{k+(\beta/p)}}=\frac{K_6}{\varphi(w)(1-|w|)^{\beta/p}}\le\frac{K_7}{\varphi(w)(1-|w|^2)^{\beta/p}}. \tag{10}$$

Combining the obtained inequalities (7)–(10) for J_1, J_2 and J_3, we get the constant M_1 such that $\displaystyle\int_D\frac{\psi(z)}{(1-|z|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}}dA(z)\le M_1\frac{1}{\varphi(w)(1-|w|^2)^{\beta/p}}$.

The Lemma is proved.

The Main Result. The following Schur test is a useful tool for estimation of L^p.

Theorem 1 (Schur test). Suppose (X,μ) is a measure space, $1<p<\infty$, and $\dfrac{1}{p}+\dfrac{1}{q}=1$. For a nonnegative kernel $H(x,y)$ consider the integral operator

$$(Tf)(x)=\int_X H(x,y)f(y)d\mu(y).$$

If there exist a positive function h on X and a positive constant C such that

$$\int_X H(x,y)h(y)^q\,d\mu(y)\le Ch(x)^q$$

for almost all $x\in X$ and

$$\int_X H(x,y)h(x)^p\,d\mu(x)\le Ch(y)^p$$

for almost all $y\in X$, then the operator T is bounded on $L^p(X,\mu)$ with $\|T\|\le C$.

Theorem 2. For $p(k-\alpha)<1$ the integral operator (2) is bounded in $L^p(D,dA)$.

Proof. According to the Schur test, it would be sufficient to find an approximation function h, for which the inequalities

$$\int_D \frac{\psi(z)\varphi(w)}{\left|1-z\overline{w}\right|^{2+\alpha}} h(w)^q dA(w) \le Mh(z)^q$$

and

$$\int_D \frac{\psi(z)\varphi(w)}{\left|1-z\overline{w}\right|^{2+\alpha}} h(z)^p dA(z) \le Mh(w)^p$$

are fulfilled. We find the function h, having $h(z)=\frac{1}{(1-|z|^2)^{\beta/pq}}$ form. That is, we should prove that

$$\int_D \frac{\varphi(w)}{(1-|w|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}} dA(w) \le M\frac{1}{\psi(z)(1-|z|^2)^{\beta/p}} \tag{11}$$

and

$$\int_D \frac{\psi(z)}{(1-|z|^2)^{\beta/p}\left|1-z\overline{w}\right|^{2+\alpha}} dA(z) \le M\frac{1}{\varphi(w)(1-|w|^2)^{\beta/p}}. \tag{12}$$

According to Lemmas 3 and 4, the inequalities (11) and (12) are valid under the conditions $k-\alpha<\frac{\beta}{p}<1+\varepsilon$, $-\varepsilon<\frac{\beta}{q}<\alpha+1-k$. Here $M=\max\{M_0,M_1\}$.

Taking into account that $q=\frac{p}{p-1}$ we rewrite these conditions as

$$(k-\alpha)p<\beta<(1+\varepsilon)p, \qquad -\varepsilon\frac{p}{p-1}<\beta<(\alpha+1-k)\frac{p}{p-1}.$$

It is only necessary to establish, at what values of p the intersection of intervals

$$\big((k-\alpha)p,\ (1+\varepsilon)p\big), \qquad \left(-\varepsilon\frac{p}{p-1},\ (\alpha+1-k)\frac{p}{p-1}\right)$$

is nonempty. As $p>1$, hence always $-\varepsilon\frac{p}{p-1}<(1+\varepsilon)p$. Therefore, it is necessary to clarify when $(\alpha+1-k)\frac{p}{p-1}>(k-\alpha)p$. From this inequality we obtain $p(k-\alpha)<1$. That is why for $p(k-\alpha)<1$ the operator (2) is bounded in $L^p(D,dA)$.

The Theorem is proved.

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