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Mathematics

DEGENERATE DIFFERENTIAL-OPERATOR EQUATIONS ON INFINITE INTERVAL

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In the present paper we consider the Dirichlet problem for the fourth order differential-operator equation $Lu \equiv (t^{\alpha}u'')'' + t^{-2}Au = f$, where $t \in (1,+\infty)$, $\alpha \geq 2$, $f \in L_{2,2}((1,+\infty),H)$, A is a linear operator in the separable Hilbert space H and has a complete system of eigenvectors that form a Riesz basis in H. The existence and uniqueness of the generalized solution for the Dirichlet problem are proved, and the description of spectrum for the corresponding operator is given.

***Keywords*:** Dirichlet problem, weighted Sobolev spaces, differential equations in abstract spaces, spectrum of the linear operator.

1. The Problem Formulation. In this paper we consider the Dirichlet problem for the fourth order differential-operator equation

$$Lu \equiv (t^{\alpha}u'')'' + t^{-2}Au = f, \tag{1}$$

where $t \in (1,+\infty)$, $\alpha \geq 2$, $f \in L_{2,2}(1,+\infty)$. We assume that the operator A has a complete system $\{\varphi_k\}_{k=1}^{\infty}$ of eigenfunctions $A\varphi_k = a_k\varphi_k$, $k \in N$, that form the Riesz basis in H, i.e. for every $x \in H$ we have a representation $x = \sum_{k=1}^{\infty} x_k\varphi_k$ and $c_1\sum_{k=1}^{\infty}|x_k|^2 \leq \|x\|^2 \leq c_2\sum_{k=1}^{\infty}|x_k|^2$. First we give the definition and some properties of the weighted Sobolev space $\dot{W}_{\alpha}^{2}(1,+\infty)$, as well give definition of the generalized solution of Dirichlet problem for one-dimensional equation (1), i.e. when the operator $Au = au$, $a \in C$, $a = \text{const}$ (see [1, 2]). Then the description of the spectrum $\sigma(L)$ of L operator is given.

Our approach is based on investigations of the one-dimensional equation. This method for the second and fourth order degenerate equations in the finite interval was applied in [3, 4] and for higher order equations in [5].

Then using the general method of A.A. Dezin (see [6]) we pass to the operator case and prove the existence and uniqueness of the generalized solution

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for equation (1) under a given condition on the operator A. Note that the operator $A: H \to H$ is, in general, unbounded.

2. One-dimensional Case.

2.1. The Space $\dot{W}_\alpha^2(1,+\infty)$**.** Denote by $\dot{W}_\alpha^2(1,+\infty)$ the completion of $\dot{C}^2[1,+\infty) = \{u \in C^2(1,+\infty), u(1) = u'(1) = u(+\infty) = u'(+\infty) = 0\}$ in the norm

$$\|u\|^2_{\dot{W}_\alpha^2(1,+\infty)} = \int_1^{+\infty} t^\alpha |u''(t)|^2 \, dt.$$

First note that for functions $u \in \dot{W}_\alpha^2(1,+\infty)$ for every $t_0 \in [1,+\infty)$ there exist the finite values $u(t_0)$, $u'(t_0)$ and $u(1) = u'(1) = 0$ (see [2, 5]).

Proposition 1. For functions $u \in \dot{W}_\alpha^2(1,+\infty)$ at large values of t we have the following estimates:

$$|u(t)|^2 \le c_1 t^{3-\alpha} \|u\|^2_{\dot{W}_\alpha^2(1,+\infty)}, \ \alpha \ne 1, \ \alpha \ne 3, \tag{2}$$

$$|u'(t)|^2 \le c_2 t^{1-\alpha} \|u\|^2_{\dot{W}_\alpha^2(1,+\infty)}, \ \alpha \ne 1. \tag{3}$$

For $\alpha = 3$ in the inequality (2) we replace $t^{3-\alpha}$ with $|\ln t|$, and for $\alpha = 1$ we replace $t^{3-\alpha}$ with $t^2 |\ln t|$ in (2) and $t^{1-\alpha}$ with $|\ln t|$ in (3).

It follows from Proposition 1 that for $\alpha > 3$ the conditions $u(+\infty) = u'(+\infty) = 0$ "are retained" after the completion, for $1 < \alpha \le 3$ only the condition $u'(+\infty) = 0$ "is retained", while for $\alpha \le 1$ the values $u(+\infty)$ and $u'(+\infty)$, in general, can become the infinity.

Let $L_{2,\beta}(1,+\infty) = \left\{ f, \|f\|^2_{L_{2,\beta}(1,+\infty)} = \int_1^\infty t^\beta |f(t)|^2 \, dt < \infty \right\}.$

Proposition 2. For $\alpha \ge 2$ we have the continuous embedding

$$\dot{W}_\alpha^2(1,+\infty) \subset L_{2,-2}(1,+\infty), \tag{4}$$

which is compact for $\alpha > 2$.

Note that the embedding (4) fails for $\alpha < 2$, and is not compact for $\alpha = 2$ (see [2]).

2.2. One-dimensional Equation. In this section we define the generalized solution of the Dirichlet problem for one-dimensional equation (1)

$$Lu \equiv (t^\alpha u'')'' + a t^{-2} u = f, \ \alpha \ge 2, \ f \in L_{2,2}(1,+\infty). \tag{5}$$

First we define the particular case of equation (5) for $a = 0$

$$Bu \equiv (t^\alpha u'')'' = f, \ \alpha \ge 2, \ f \in L_{2,2}(1,+\infty). \tag{6}$$

Definition 1. The function $u \in \dot{W}_\alpha^2(1,+\infty)$ is called the generalized solution of the Dirichlet problem for equation (6), if for every $v \in \dot{W}_\alpha^2(1,+\infty)$ we have

$$(t^\alpha u'', v'') = (f, v),$$

where $(\cdot,\cdot)$ stands for the scalar product in $L_2(1,+\infty)$.

Proposition 3. The generalized solution of the equation (6) exists and is unique for every $f \in L_{2,2}(1,+\infty)$.

Denote by $B: D(B) \subset \dot{W}_\alpha^2(1,+\infty) \subset L_{2,-2}(1,+\infty) \to L_{2,2}(1,+\infty)$ the operator, corresponding to Definition 1.

Proposition 4. The domain of definition of B operator consists of functions $u \in \dot{W}_\alpha^2(1,+\infty)$, for which $u'(+\infty)=0$ and the value $u(+\infty)$ is finite for $\alpha > \frac{5}{2}$ ($u(+\infty)$ can not be arbitrarily, but is determined by the right-hand side of equation (6)).

So, from the classical point of view the correct formulation of problem (6) (as well as for problem (5), since $D(L)=D(B)$) is:

for $\alpha > 3$ $\quad u(1)=u'(1)=u(+\infty)=u'(+\infty)=0,$

for $\frac{5}{2} < \alpha \le 3$ $\quad u(1)=u'(1)=u'(+\infty)=0$ and $u(+\infty)$ is finite,

for $2 \le \alpha \le \frac{5}{2}$ $\quad u(1)=u'(1)=u'(+\infty)=0$.

The operator B acts from $L_{2,-2}(1,+\infty)$ into $L_{2,2}(1,+\infty)$. To have an operator acting in the same space, which is necessary for consideration of spectral problems, denote by $Bu = t^2 Bu,\ D(B)=D(B)$. Since for $g(t)=t^2 f(t)$ we have $\| g \|_{L_{2,-2}(1,+\infty)} = \| f \|_{L_{2,2}(1,+\infty)}$, hence we get an operator in $L_{2,-2}(1,+\infty)$, i.e. $B: L_{2,-2}(1,+\infty) \to L_{2,-2}(1,+\infty)$.

Theorem 1. The operator $B: L_{2,-2}(1,+\infty) \to L_{2,-2}(1,+\infty)$ is positive and self-adjoint for $\alpha \ge 2$. The inverse operator $B^{-1}: L_{2,-2}(1,+\infty) \to L_{2,-2}(1,+\infty)$ is bounded and is compact for $\alpha > 2$. The spectrum of operator B for $\alpha > 2$ is discrete $\sigma(B)=\sigma_p(B)$, and for $\alpha = 2$ is purely continuous $\sigma(B)=\sigma_c(B)$. It coincides with the ray $\left[\frac{1}{16}, +\infty\right)$.

Note that now we can rewrite equation (5) in the form $Bu = -au + g$, i.e. we can consider the number $-a$ as a spectral parameter (see [2]) and, hence, for $-a \in \rho(B)$, where $\rho(B)$ is the set of regular points for operator B, equation (5) is uniquely solvable for every $f \in L_{2,2}(1,+\infty)$ $(g \in L_{2,-2}(1,+\infty))$. Note also that $D(L)=D(B)=D(B)$.

3. Operator Equation. Now consider the operator equation (1). First describe a special class of the so-called $\prod$-operators A, which have complete system $\{\varphi_k\}_{k=1}^\infty$ of eigenfunctions $A\phi_k = a_k \phi_k,\ k \in N,$ forming a Riesz base in H (see [6]). Let V_x be n-dimensional cube in R^n with the side 2π. Denote by P^∞ the linear manifold of infinitely differentiable complex functions that are periodical with respect to all variables $(x_1, x_2, \dots, x_n) \in \overline{V}_x$ having the period 2π. We associate to the polynomial $A(s),\ s \in Z^n$, with constant complex coefficients

$$A(s)=\sum_{|\alpha| \le m} a_\alpha s^\alpha,\ \alpha=(\alpha_1,\alpha_2,\dots,\alpha_n) \in Z_+^n,\ s^\alpha = s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n},\ |\alpha| = \alpha_1+\alpha_2+\dots+\alpha_n,$$

the differential operation $A(-iD_x)$ such that $A(-iD_x)e^{is\cdot x}=A(s)e^{is\cdot x}$, $s\cdot x=s_1x_1+s_2x_2+\cdots+s_nx_n$. Now define the operator $A:L_2(V_x)\to L_2(V_x)$ as the closure of differential operation $A(-iD_x)$, originally given on P^∞. It is evident that the system of the eigenfunctions $e^{is\cdot x}$, $s\in Z^n$, forms an orthogonal base in $L_2(V_x)$, corresponding to the eigenvalues $A(S)=\{A(s),s\in Z^n\}$. We have the following important

Proposition 5. The spectrum $\sigma(A)$ of operator A coincides with the closure $\overline{A(S)}$ in C of the set $A(S)$, which forms a point spectrum $\sigma_p(A)$ of operator A. The continuous spectrum $\sigma_c(A)$ coincides with $\overline{A(S)}\setminus A(S)$.

Note that for $n\le 2$ the resolvent set $\rho(A)$ for the $\prod$-operator A is always nonempty, but for $n\ge 2$ the spectrum $\sigma(A)$ can fill the whole complex plane C. For example, if $n=3$, then for the polynomial $A(s)=s_1+\alpha s_2+i(s_3+\beta s_2^2)$, where α,β are irrational numbers, the set $A(S)$ is dense in C.

Since the system $\{\varphi_k\}_{k=1}^\infty$ forms a base in H, we can write $u(t)=\sum_{k=1}^\infty u_k(t)\varphi_k$, $f(t)=\sum_{k=1}^\infty f_k(t)\varphi_k$. Then the operator equation (1) is decomposed to an infinite chain of ordinary differential equations

$$L_ku_k\equiv(t^\alpha u_k^{(m)})^{(m)}+a_kt^{-2}u_k=f_k,\ k\in N. \tag{7}$$

From $f\in L_{2,2}((1,+\infty),H)$ it follows that $f_k\in L_{2,2}(1,+\infty)$, $k\in N$. Hence we get the equation (5), for which we discussed the existence and uniqueness of the generalized solution in the subsection 2.2.

Definition 2. The function $u\in L_{2,-2}((1,+\infty),H)$ is called the generalized solution for equation (1), if in representation $u(t)=\sum_{k=1}^\infty u_k(t)\varphi_k$ the functions $u_k(t)\in \dot W_\alpha^2(1,+\infty),k\in N,$ are generalized solutions of Dirichlet problem for equations (7) (see Definition 1).

Actually the operator L is defined as the closure in $L_{2,-2}((1,+\infty),H)$ of the differential expression $L(D)$, originally given on all finite linear combinations of functions $u_k(t)\varphi_k$, $k\in N$, where $u_k\in D(L_k)$.

Of great importance at the transition to the operator case is the following (see [6])

Proposition 6. The operator equation (1) is uniquely solvable for every $f\in L_{2,2}((1,+\infty),H)$, iff the equations (11) are uniquely solvable for every $f_k\in L_{2,2}(1,+\infty)$, $k\in N$, and the inequalities

$$\|u_k\|_{L_{2,-2}(1,+\infty)}\le c\|f_k\|_{L_{2,2}(1,+\infty)} \tag{8}$$

are fulfilled uniformly with respect to $k\in N$.

Proof. The Necessity. Let the operator equation (1) be uniquely solvable. If for some $k \in N$ the unique solvability of equation (7) breaks and $u_k(t) \in W_\alpha^2(1,+\infty)$ is a nontrivial solution for homogeneous equation, then the function $u_k(t)\varphi_k$ will be the nontrivial solution for the homogeneous equation (1). If equation (7) for all $k \in N$ is uniquely solvable for every right-hand side, but fails (8), then there exists a sequence $f_{k_m} \in L_{2,2}(1,+\infty)$, $m \in N$, such that

$$\|u_{k_m}\|_{L_{2,-2}(1,+\infty)} \geq m \|f_{k_m}\|_{L_{2,2}(1,+\infty)}, \; m \in N. \tag{9}$$

Then, the inverse operator L^{-1} does exist, it is given on the dense set (on finite linear combinations of the functions $f_k(t)\varphi_k$, $f_k \in L_{2,2}(1,+\infty)$), but is an unbounded operator (it suffice to consider the right-hand sides of functions $f_{k_m}(t)\varphi_{k_m}$, $f_{k_m} \in L_{2,2}(1,+\infty)$ and use inequalities (9)). Since the operator L^{-1} is closed, then $D(L^{-1})$ can not coincide with the whole space $L_{2,2}((1,+\infty),H)$.

The Sufficiency. Let the equations (7) are uniquely solvable for every $f_k \in L_{2,2}(1,+\infty)$, $k \in N$, and inequalities (8) are fulfilled uniformly with respect to $k \in N$. Then the function $u(t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k$, where the functions $u_k(t)$, $k \in N$, are the solutions of equations (7), is the solution of the operator equation (1). Indeed, since the system $\{\varphi_k\}_{k=1}^{\infty}$ is the Riesz base, we can write

$$\|u\|^2_{L_{2,-2}((1,+\infty),H)} = \int_1^{+\infty} t^{-2} \|u(t)\|_H^2 \, dt \leq c_2 \int_1^{+\infty} t^{-2} \sum_{k=1}^{\infty} |u_k(t)|^2 dt \leq c_2 c \sum_{k=1}^{\infty} \|f_k\|^2_{L_{2,2}(1,+\infty)} \leq c_3 \|f\|^2_{L_{2,2}((1,+\infty),H)},$$

where $f(t) = \sum_{k=1}^{\infty} f_k(t)\varphi_k$, i.e. we have $\|u\|_{L_{2,-2}((1,+\infty),H)} \leq c_4 \|f\|_{L_{2,2}((1,+\infty),H)} =$ $= c_4 \|Lu\|_{L_{2,2}((1,+\infty),H)}$, and, therefore, the operator L^{-1} is defined on the whole space $D(L^{-1}) = L_{2,2}((1,+\infty),H)$ and is bounded.

The proof is complete.

Note that the inverse operator $L^{-1}: L_{2,2}((1,+\infty),H) \to L_{2,-2}((1,+\infty),H)$ for $\alpha > 2$ is compact only in the case, when H is finite-dimensional.

Suppose that the following conditions are valid

$$\rho(-a_k, \sigma(B)) > \varepsilon, \; k \in N, \tag{10}$$

where $\varepsilon > 0$, and ρ is the distance in the complex plane.

Theorem 2. Under the conditions (10) the generalized solution of equation (1) exists and is unique for every $f \in L_{2,2}((1,+\infty),H)$.

For the proof first note that under conditions (10) the equations (7) are uniquely solvable and inequalities (8) are fulfilled. It remains only to use Proposition 6.

Denote by $\tilde{L} = t^2 L$, $D(\tilde{L}) = D(L)$. If the operator A is self-adjoint, then the following description of spectrum for the operator $\tilde{L}: L_{2,-2}((1,+\infty),H) \to L_{2,-2}((1,+\infty),H)$ may be given.

Theorem 3. The spectrum of operator $\tilde{L}$ coincides with the closure of the direct sum $\sigma(B)$ and $\sigma(A)$, i.e. $\sigma(\tilde{L}) = \overline{\sigma(B)+\sigma(A)} \equiv \overline{\{\lambda_1+\lambda_2 : \lambda_1 \in \sigma(B), \lambda_2 \in \sigma(A)\}}$.

The proof follows from the representation of operator $\tilde{L}$ in the form

$$\tilde{L} = B \otimes I_H + I_{L_{2,-2}(1,+\infty)} \otimes A,$$

where $\otimes$ denotes the tensor product of the operators (see [7]).

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REFERENCES

1. **Kudryavtzev L.D.** Trudy Mat. Inst. AN SSSR, 1984, v. 170, p. 161–190 (in Russian).
2. **Tepoyan L.P, Hosein Ansari** Vestnik RAU. Physical-Mathematical and Natural Sciences, 2010, № 1, p. 3–10 (in Russian).
3. **Dezin A.A.** Mat. Sbornik, 1982, v. 43, № 3, p. 287–298 (in Russian).
4. **Tepoyan L.P** Differentcialnye Uravnenia, 1987, v. 23, № 8, p. 1366–1376 (in Russian); English transl. in Amer. Math. Society, 1988, v. 23, № 8, p. 930– 939.
5. **Tepoyan L.P** Izv. NAN Armenii. Matematika, 1999, v. 34, № 5, p. 48–56.
6. **Dezin A.A.** Partial Differential Equations (An Introduction to a General Theory of Linear Boundary Value Problems). Springer, 1987.
7. **Berezanski J.M.** Expansion in Eigenfunctions of Selfadjoint Operators. Transl. Math. Monographs 17. American Mathematical Society. Providence, 1968.