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Mathematics

S-UNIVERSALITY IN 2-CATEGORIES

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For 2-categories the notion of S-universality is introduced and investigated.

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"The bold step of really formulating the general notion of a universal arrow was taken by Samuel [1] in 1948; the general notion was then lavishly popularized by Bourbaki [2]". In order to interpret Grothendieck's extension of the fundamental theorem of Galois theory [3] in abstract categories more general notions than the universality is required [4]. Here this notion of S-universality for 2-categories is introduced and investigated. Firstly, we give the definitions of 2-categories, 2-functors and 2-transformations in a form, which is convenient for our goals. For comparison see [5–7].

1. A 2-category $\mathscr{C}$ consists of a date base, satisfying some system of axioms. The date base for 2-categories is defined as follows:

(i) an arbitrary set of objects $\mathscr{C}_0 = \text{Ob}\,\mathscr{C}$;

(ii) sets $\mathscr{C}(\mathscr{X}, \mathscr{Y})_0 = \text{Mor}_{\mathscr{C}}(\mathscr{X}, \mathscr{Y})$ of morphisms $\mathscr{X}\, F\, \mathscr{Y}$ from $\mathscr{X}$ to $\mathscr{Y}$ for all $\mathscr{X}, \mathscr{Y} \in \text{Ob}\,\mathscr{C}$;

(iii) sets $\mathscr{C}(F, G) = \text{Trans}\,(F, G)$ of transformations (or morphisms of rank 2) $F\,\xi\,G$ for all $F, G \in \mathscr{C}(\mathscr{X}, \mathscr{Y})_0$ and for all $\mathscr{X}, \mathscr{Y} \in \text{Ob}\,\mathscr{C}$;

(cm) a composite map for morphisms $F \in \mathscr{C}(\mathscr{X}, \mathscr{Y})_0$ and $K \in \mathscr{C}(\mathscr{Y}, \mathscr{Z})_0$ with $F \circ K \in \mathscr{C}(\mathscr{X}, \mathscr{Z})_0$, where $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \text{Ob}\,\mathscr{C}$;

(ct) a composite map for transformations $\xi \in \mathscr{C}(F, G)$ and $\eta \in \mathscr{C}(G, H)$ with $\xi \circ \eta \in \mathscr{C}(F, H)$, where F, G, H run $\mathscr{C}(\mathscr{X}, \mathscr{Y})_0$ and $\mathscr{X}, \mathscr{Y}$ run $\text{Ob}\,\mathscr{C}$;

(mt) a multiplication map for transformations $\xi \in \mathscr{C}(F, G)$ and $\rho \in \mathscr{C}(K, L)$ with $\xi \cdot \rho \in \mathscr{C}(F \circ K, G \circ L)$, where F, G run $\mathscr{C}(\mathscr{X}, \mathscr{Y})_0$, K, L run $\mathscr{C}(\mathscr{Y}, \mathscr{Z})_0$ and $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \text{Ob}\,\mathscr{C}$;

(im) a system of identity morphisms $e(\mathscr{X}) = 1_{\mathscr{X}} \in \mathscr{C}(\mathscr{X}, \mathscr{Y})_0$ for any $\mathscr{X} \in \text{Ob}\,\mathscr{C}$;

(it) for any $F \in \mathscr{C}(\mathscr{X}, \mathscr{Y})_0$ and any $\mathscr{X}, \mathscr{Y} \in \text{Ob}\,\mathscr{C}$ a system of identity transformations $1_F \in \mathscr{C}(F, G)$.

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This date base of the 2-category must satisfy the following system of axioms:
1) $F \circ (K \circ R) = (F \circ K) \circ R$; 2) $1_{\mathcal{X}} \circ F = F = F \circ 1_{\mathcal{Y}}$; 3) $\xi \circ (\eta \circ \zeta) = (\xi \circ \eta) \circ \zeta$;
4) $1_{F} \circ \xi = \xi = \xi \circ 1_G$; 5) $\xi \bullet (\rho \bullet \alpha) = (\xi \bullet \rho) \bullet \alpha$; 6) $1_{e(\mathcal{X})} \bullet \xi = \xi \bullet 1_{e(\mathcal{Y})}$;
7) $(\xi \circ \eta) \bullet (\rho \circ \sigma) = (\xi \bullet \rho) \circ (\eta \bullet \sigma)$; 8) $1_F \bullet 1_K = 1_{F \circ K}$,
for transformations $\xi \in \mathcal{C}(F, G)$, $\eta \in \mathcal{C}(G, H)$, $\zeta \in \mathcal{C}(H, A)$, $\rho \in \mathcal{C}(K, L)$, $\sigma \in \mathcal{C}(L, M)$, $\alpha \in \mathcal{C}(R, S)$, for morphisms $F, G, H, A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$, $K, L, M \in \mathcal{C}(\mathcal{Y}, \mathcal{Z})_0$, $R, S \in \mathcal{C}(\mathcal{Z}, \mathcal{W})_0$ and objects $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \in \mathrm{Ob}\,\mathcal{C}$.

The system of all morphisms of the 2-category $\mathcal{C}$ is denoted by $\mathrm{Mor}_{\mathcal{C}}$ or $\mathcal{C}_M$, the system of all transformations of $\mathcal{C}$ is denoted by $\mathrm{Trans}_{\mathcal{C}}$ or $\mathcal{C}_T$. If $\mathcal{C}$ is the 2-category, then for any objects $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{C}$ the sets $\mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and $\mathcal{C}(F, G)$ for all $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ constitute a usual category (or 1-category), which is denoted as $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. A well-known example of a 2-category is the 2-category **Cat** of categories, covariant functors and natural transformations.

2. Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary 2-categories. A 2-functor Φ from $\mathcal{C}$ to $\mathcal{D}$ is given by

(i) a map $\Phi_0\colon \mathcal{C}_0 \to \mathcal{D}_0$;

(ii) maps $\Phi_{\mathcal{X}\mathcal{Y}}\colon \mathcal{C}(\mathcal{X}, \mathcal{Y})_0 \to \mathcal{D}(\mathcal{X}\Phi_0, \mathcal{Y}\Phi_0)_0$ for all $\mathcal{X}, \mathcal{Y} \in \mathrm{Ob}\,\mathcal{C}$;

(iii) maps $\Phi_{FG}\colon \mathcal{C}(F, G) \to \mathcal{D}(F\Phi_{\mathcal{X}\mathcal{Y}}, \mathcal{G}\Phi_{\mathcal{X}\mathcal{Y}})$ for all $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$ and $\mathcal{X}, \mathcal{Y} \in \mathrm{Ob}\,\mathcal{C}$,

which satisfy the following conditions:

1) $(F \circ K)\, \Phi_{\mathcal{X}\mathcal{Z}} = F\, \Phi_{\mathcal{X}\mathcal{Y}} \circ K\, \Phi_{\mathcal{Y}\mathcal{Z}}$; 2) $e(\mathcal{X})\, \Phi_{\mathcal{X}\mathcal{X}} = e(\mathcal{X}\Phi_0)$;

3) $(\xi \circ \eta)\, \Phi_{FH} = (\xi\, \Phi_{FG}) \circ (\eta\, \Phi_{GH})$; 4) $e(F)\, \Phi_{FF} = e(F\, \Phi_{\mathcal{X}\mathcal{Y}})$;

5) $(\xi \bullet \rho)\, \Phi_{(F \circ K)(G \circ L)} = \xi\, \Phi_{FG} \bullet \rho\, \Phi_{KL}$.

Remark. The condition of naturality in Gray [8] follows from the conditions 2) and 5). Further the subscripts of the 2-functor are often omitted.

3. A natural 2-transformation $\boldsymbol{J}$ of a 2-functor $\mathcal{C}\,\Phi\,\mathcal{D}$ into a 2-functor $\mathcal{C}\,\Psi\,\mathcal{D}$ is a map $\boldsymbol{J}_0 : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ such that

(a0) $\boldsymbol{J}_0(\mathcal{X})$ is a morphism $(\mathcal{X}\,\Phi)\, \boldsymbol{J}_{\mathcal{X}}\, (\mathcal{X}\,\Psi)$ for any $\mathcal{X} \in \mathrm{Ob}\,\mathcal{C}$;

(a1) $F\Phi \circ \boldsymbol{J}_{\mathcal{Y}} = \boldsymbol{J}_{\mathcal{X}} \circ F\Psi$ for any $\mathcal{X}\,F\,\mathcal{Y} \in \mathrm{Mor}\,\mathcal{C}$;

(a2) $\xi\Phi \bullet e(\boldsymbol{J}_{\mathcal{Y}}) = e(\boldsymbol{J}_{\mathcal{X}}) \bullet \xi\Psi$ for any $(F\,\xi\,G) \in \mathrm{Trans}\,\mathcal{C}$, F and $\in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$.

Then one can define a map $\boldsymbol{J}_T$: Trans $\mathcal{C} \to$ Trans $\mathcal{D}$ such that $\boldsymbol{J}_T\,(F\,\xi\,G)$, where $F, G \in \mathcal{C}(\mathcal{X}, \mathcal{Y})_0$, is the transformation $\boldsymbol{J}_{\xi} = \xi\Phi \bullet e(\boldsymbol{J}_{\mathcal{Y}}) = e(\boldsymbol{J}_{\mathcal{X}}) \bullet \xi\Psi$ of the functor $F\Phi \circ \boldsymbol{J}_{\mathcal{Y}} = \boldsymbol{J}_{\mathcal{X}} \circ F\Psi$ into the functor $G\Phi \circ \boldsymbol{J}_{\mathcal{Y}} = \boldsymbol{J}_{\mathcal{X}} \circ G\Psi$, which both are functors from $\mathcal{X}\Phi$ to $\mathcal{Y}\Psi$.

This map satisfies the following equalities:

(b1) $\boldsymbol{J}_T\,(1_F) = e(F\Phi \circ \boldsymbol{J}_{\mathcal{Y}}) = e(\boldsymbol{J}_{\mathcal{X}}) \circ F\Psi$;

(b2) $\boldsymbol{J}_T\,((F\,\xi\,G) \circ (G\,\eta\,H)) = \boldsymbol{J}_T\,(\xi) \circ \boldsymbol{J}_T\,(\eta)$;

(b3) $\boldsymbol{J}_T\,((F\,\xi\,G) \bullet (K\,\rho\,L)) = \boldsymbol{J}_T\,(\xi) \bullet \rho\Psi = \xi\Phi \bullet \boldsymbol{J}_T\,(\rho)$

for any $F, G, H \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$, for any $K, L \in \mathcal{C}(\mathcal{Y}, \mathcal{Z})$ and for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathrm{Ob}\,\mathcal{C}$.

Further, the notion of S-universal morphisms is introduced and investigated.

4. Let S_M be a property of morphisms, S_T be a property of transformations. Usually we suppose that the properties S are

(u) unitary: any identity element (morphism or transformation) has the property S;

(m) multiplicative: the composite (resp., product) of morphisms or transformations has the property S, if each composant has this property.

Naturally, any property S for an element x can be expressed in the form $x \in \mathscr{M}_S$, where $\mathscr{M}_S$ is the set of all elements satisfying the property S.

5. *Definition.* Let $\mathscr{C}$ and $\mathscr{D}$ be arbitrary 2-categories, S_M and S_T be arbitrary properties of morphisms and transformations respectively. Let $\mathscr{D} u \mathscr{C}$ be a 2-functor and $\mathscr{X}$ be an object in $\mathscr{C}$. Then a pair ($\underline{\mathscr{X}}$; $\mathscr{X}\ J_{\mathscr{X}}\ (\underline{\mathscr{X}}u)$), consisting of an object $\underline{\mathscr{X}}$ in $\mathscr{D}$ and a morphism $J_{\mathscr{X}}$ in $\mathscr{C}$, is called S-universal pair from $\mathscr{X}$ to u, if

(i) for any object $\mathscr{P}$ in $\mathscr{D}$ and for any morphism $\mathscr{X}\ F\ (\mathscr{P}u)$ in $\mathscr{C}$ there is a unique morphism $\underline{\mathscr{X}}\ \underline{F}\ \mathscr{P}$ with property S_M such that $J_{\mathscr{X}} \circ \underline{F}u = F$;

(ii) for any two morphisms F and G from $\mathscr{X}$ to $\mathscr{P}u$ and for any transformation $F\ \xi\ G$ there is the only transformation $\underline{F}\ \underline{\xi}\ \underline{G}$ with the property S_T and such that $e(J_{\mathscr{P}} \cdot \underline{\xi}u) = \xi$.

Dually, the S-couniversality is defined. Note, that if all morphisms and transformations in $\mathscr{D}$ satisfy respectively the properties S_M and S_T, we get the usual definition of (co)universality.

Let ($\underline{\mathscr{X}}$; $\mathscr{X}\ J_{\mathscr{X}}\ (\underline{\mathscr{X}}u)$) and ($\underline{\mathscr{Y}}$, $\mathscr{Y}\ J_{\mathscr{Y}}\ (\underline{\mathscr{Y}}u)$) be two S-universal pairs respectively from $\mathscr{X}$ and $\mathscr{Y}$ to $\mathscr{C}u\mathscr{D}$. A morphism from ($\underline{\mathscr{X}}$; $\mathscr{X}\ J_{\mathscr{X}}\ (\underline{\mathscr{X}}u)$) to ($\underline{\mathscr{Y}}$, $\mathscr{Y}\ J_{\mathscr{Y}}(\underline{\mathscr{Y}}u)$) is a pair of morphisms $\mathscr{X}\ K\ \mathscr{Y}$ in $\mathscr{C}$ and $\underline{\mathscr{X}}\ \underline{K}\ \underline{\mathscr{Y}}$ in $\mathscr{D}$ such that $J_{\mathscr{X}} \circ \underline{K}u = K \circ J_{\mathscr{Y}}$. Emphasize that this condition determines $\underline{K}$ uniquely.

If (K, $\underline{K}$) and (L, $\underline{L}$) are two morphisms from ($\underline{\mathscr{X}}$; $J_{\mathscr{X}}$) to ($\underline{\mathscr{Y}}$, $J_{\mathscr{Y}}$), then the transformation from (K, $\underline{K}$) to (L, $\underline{L}$) is a pair of transformations $K\rho L$ in $\mathscr{C}$ and $\underline{K}\ \underline{\rho}\ \underline{L}$ in $\mathscr{D}$, where the last transformation is uniquely determined by condition $\rho \cdot e(J_{\mathscr{Y}}) = e(J_{\mathscr{X}}) \cdot \underline{\rho}$.

All S-universal pairs from objects in $\mathscr{C}$ to the functor $\mathscr{D}u\mathscr{C}$, their morphisms and the transformations of morphisms together with composition of morphisms, composition and multiplication of transformations, identity morphisms and identity transformations, that are induced by those in $\mathscr{C}$ and $\mathscr{D}$, from a 2-category ($\mathscr{C}$, u, S) of S-universal pairs from objects in $\mathscr{C}$ to u.

Dually, the category (v, $\mathscr{D}$, S^*) of S-couniversal pairs from a 2-functor $\mathscr{C}v\mathscr{D}$ to objects in $\mathscr{D}$ is defined.

6. *Proposition.* If the properties S_M and S_T are unitary and multiplicative both for the composites and for the products, then for any object $\mathscr{X}$ in $\mathscr{C}$ and any 2-functor $\mathscr{D}\ u\ \mathscr{C}$ the S-universal pair ($\underline{\mathscr{X}}$; $J_{\mathscr{X}}$) from $\mathscr{X}$ to u is determined uniquely up to the isomorphism of the object $\underline{\mathscr{X}}$. More exactly,

(i) if ($\underline{\mathscr{X}}'J_{\mathscr{X}'}$) is another S-universal pair from $\mathscr{X}$ to u and S_M is multiplicative for composites and unitary, then the only morphism $\underline{\mathscr{X}}\ \underline{J_{\mathscr{X}'}}\ \underline{\mathscr{X}}'$, satisfying S_M and $J_{\mathscr{X}} \circ \underline{J_{\mathscr{X}'}}\,u = J_{\mathscr{X}'}$, is the isomorphism;

(ii) if $\underline{\mathscr{X}}\ \underline{J}\ \underline{\mathscr{X}}'$ is an isomorphism such that $\underline{J}$, $\underline{J}^{-1}$, $e(\underline{J})$, $e(\underline{J}^{-1})$ have the properties S_M and S_T respectively, which are multiplicative for composites of morphisms and products of transformations, then $J_{\mathscr{X}'} = J_{\mathscr{X}} \circ \underline{J}$ together with $\underline{\mathscr{X}}'$ constitutes a S-universal pair from $\mathscr{X}$ to u.

7. *Proposition.* Consider an arbitrary object $\underline{\mathscr{X}}$ in $\mathscr{D}$, an object $\mathscr{X}$ and a morphism $\mathscr{X} J \underline{\mathscr{X}}u$ in $\mathscr{C}$. Then for any object $\mathscr{P}$ in $\mathscr{D}$ the maps

$$\varphi_M: \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})_0 \to \mathscr{C}(\mathscr{X}, \mathscr{P}u)_0, \quad \underline{F} \mapsto J \circ Fu,$$

$$\varphi_T: \mathscr{D}(\underline{F}, \underline{G}) \to \mathscr{C}(\underline{F}\varphi_M, \underline{G}\varphi_M), \quad \xi \Vert\!\!\to 1_J \cdot \xi \text{ for all } \underline{F}, \underline{G} \in \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})_0$$

constitute a 1-functor of 1-categories $\varphi: \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P}) \to \mathscr{C}(\mathscr{X}, \mathscr{P}u)$.

Dually, a functor $\varphi^*: \mathscr{C}(\mathscr{X}, \underline{\mathscr{P}}) \to \mathscr{D}(\mathscr{X}v, \mathscr{P})$ is determined.

8. *Proposition.*

(a) The pair $(\underline{\mathscr{X}}, \mathscr{X} J_{\mathscr{X}} (\underline{\mathscr{X}}u))$ is a S-universal pair from $\mathscr{X}$ to u, if and only if the composants of the functor $\varphi = (\varphi_M, \varphi_T)$ associated with J have sections, i.e. left inverse maps

$$\psi_M: \mathscr{C}(\mathscr{X}, \mathscr{P}u)_0 \to \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})_0,$$

$$\psi_T: \mathscr{C}(\underline{F}\varphi_M, \underline{G}\varphi_M) \to \mathscr{D}(\underline{F}, \underline{G}), \quad \underline{F}, \underline{G} \in \mathscr{D}, \ \underline{\mathscr{X}} \in \mathscr{P}$$

for all $\mathscr{P} \in \mathscr{D}_0$.

Moreover, one can take

(i) the property S_M for a functor $\underline{F}$ is $\underline{F} \in \operatorname{Im} \psi_M$;

(ii) the property S_T for a transformation $\underline{F}\ \xi\ \underline{G}$ with $\underline{F}, \underline{G} \in \underline{\mathscr{D}}(\underline{\mathscr{X}}, \mathscr{P})$ is $\xi \in \operatorname{Im} \psi_T$.

(b) The pair $\psi = (\psi_M, \psi_T)$ gives a functor from $\mathscr{C}(\mathscr{X}, \mathscr{P}u)$ to $\mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})$, if and only if S_T is a unitary and multiplicative for composites property.

A dual necessary and sufficient condition is true for a S-couniversal pair $(\underline{\mathscr{P}}, \underline{\mathscr{P}}v)$ I $\mathscr{P}$ from v to $\mathscr{P}$ and the pair of sections

$$\psi_M^*: \mathscr{D}(\mathscr{X}v, \mathscr{P})_0 \to \mathscr{C}(\mathscr{X}, \underline{\mathscr{P}})_0,$$

$$\psi_T^*: \mathscr{D}(\underline{U}\varphi_M^*, \underline{V}\varphi M^*)O_0 \to \mathscr{C}(U, V)_0, \quad U, V \in \mathscr{C}(\mathscr{X}, \underline{\mathscr{P}})_0$$

gives a functor from $\mathscr{D}(\mathscr{X}v, \mathscr{P})$ to $\mathscr{C}(\mathscr{X}, \underline{\mathscr{P}})$, if and only if S_T^* is a multiplicative for composite and unitary property.

9. Let $\mathscr{D}\ u\ \mathscr{C}$ be an arbitrary 2-functor, $\underline{\mathscr{X}}$ and $\mathscr{X}$ be arbitrary objects in $\mathscr{D}$ and $\mathscr{C}$ respectively.

Proposition. By assigning to each

(i) object $\mathscr{P} \in \mathscr{D}_0$ the objects $\mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})$ and $\mathscr{C}(\mathscr{X}, \mathscr{P}u)$;

(ii) morphism $\mathscr{P}\ U\ \mathscr{Q}$ in $\mathscr{D}$ the morphisms.

(a) $\mathscr{D}(\underline{\mathscr{X}}, U): \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P}) \to \mathscr{D}(\underline{\mathscr{X}}, \mathscr{Q})$ such that $\underline{F} \mapsto \underline{F} \circ U$, $\underline{F}\ \xi\ \underline{G} \mapsto (\underline{F} \circ U)(\xi \cdot 1_U)(\underline{G} \circ U)$;

(b) $\mathscr{C}(\mathscr{X}, Uu): \mathscr{C}(\mathscr{X}, \mathscr{P}u) \to \mathscr{C}(\mathscr{X}, \mathscr{Q}u)$ such that $F \mapsto F \circ Uu$,

$F\ \xi\ G \mapsto (F \circ Uu)(\xi \cdot 1_{Uu})(G \circ Uu)$.

(iii) transformation $U\ \kappa\ V$ with $U, V \in \mathscr{D}(\mathscr{P}, \mathscr{Q})$ the transformations

(a) $\mathscr{D}(\underline{\mathscr{X}}, \kappa): \mathscr{D}(\underline{\mathscr{X}}, U) \to \mathscr{D}(\underline{\mathscr{X}}, V)$ such that for any $\underline{F} \in \mathscr{D}(\underline{\mathscr{X}}, \mathscr{P})$ $\mathscr{D}(\underline{\mathscr{X}}, \kappa)_{\underline{F}} = 1_{\underline{F}} \cdot \kappa: \underline{F} \circ U \mapsto \underline{F} \circ V$;

(b) $\mathscr{C}(\mathscr{X}, \kappa u): \mathscr{C}(\mathscr{X}, Uu) \to \mathscr{C}(\mathscr{X}, Vu)$ such that for any $F \in \mathscr{C}(\mathscr{X}, \mathscr{P}u)$ $\mathscr{C}(\mathscr{X}, \kappa u)_F = = 1_F \cdot \kappa u: F \circ Uu \to F \circ Vu$, we define 2-functors $\mathscr{D}\{(\underline{\mathscr{X}}, \cdot)$ and $\mathscr{C}(\mathscr{X}, \cdot u)$ from $\mathscr{D}$ to **Cat**.

Dually, the 2-functors $\mathscr{D}(\cdot v, \mathscr{P})$ and $\mathscr{C}(\cdot, \underline{\mathscr{P}})$ from $\mathscr{C}^{op}$ to **Cat** are defined. So, if $\mathscr{C} v \mathscr{D}$ and $\mathscr{D} u \mathscr{C}$ are arbitrary 2-functors, then we have two 2-bifunctors $\mathscr{D}(\cdot v, \cdot)$ and $\mathscr{C}(\cdot, \underline{\cdot U})$ from $\mathscr{C}^{op} \times \mathscr{D}$ to **Cat.**

10. *Proposition.*

(a) Let $\mathcal{D}\, u\, \mathcal{C}$ be an arbitrary 2-functor, $\mathcal{X}$ and $\underline{\mathcal{X}}$ be arbitrary objects in $\mathcal{C}$ and $\mathcal{D}$ respectively, $\mathcal{X}\, J\, (\underline{\mathcal{X}}\, u)$ be an arbitrary morphism. Then the system of functors φ for all $\mathcal{P} \in \mathcal{D}_0$ constitutes a natural 2-transformation φ: $\mathcal{D}(\underline{\mathcal{X}}, \cdot\,) \to$ $\to \mathcal{C}(\mathcal{X}, \cdot\, u)$ of 2-functors from $\mathcal{D}$ to **Cat**.

(b) If $(\underline{\mathcal{X}}, \mathcal{X}\, J\, (\underline{\mathcal{X}}\, u))$ is a S-universal pair from $\mathcal{X}$ to u, then the system of pairs $\psi = (\psi_M, \psi_T)$ for all $\mathcal{P} \in \mathcal{D}_0$ is natural in the second variable. It gives a natural 2-transformation $\psi : \mathcal{C}(\mathcal{X}, \cdot\, u) \to \mathcal{D}(\underline{\mathcal{X}}, \cdot)$ of 2-functors from $\mathcal{D}$ to **Cat**, if the property S_T is unitary and multiplicative for composites.

11. ***Theorem.*** Given a 2-functor $\mathcal{D}\, u\, \mathcal{C}$, objects $\mathcal{X}$ in $\mathcal{C}$ and $\underline{\mathcal{X}}$ in $\mathcal{D}$, then there exists a canonical bijection between all

(i) S-universal pairs $(\underline{\mathcal{X}}, \mathcal{X}\, J_{\mathcal{X}}\, (\underline{\mathcal{X}}\, u))$ with a unitary and multiplicative for composites property S_T;

(ii) pairs (φ, ψ), consisting of natural 2-transformations

$$\varphi : \mathcal{D}(\underline{\mathcal{X}}, \cdot\,) \to \mathcal{C}(\mathcal{X}, \cdot\, u), \quad \psi : \mathcal{C}(\mathcal{X}, \cdot\, u) \to \mathcal{D}(\underline{\mathcal{X}}, \cdot\, u),$$

of 2-functors from $\mathcal{D}$ to **Cat**, satisfying the condition $\psi \circ \varphi = 1$.

Dually, given a 2-functor $\mathcal{C}\, v\, \mathcal{D}$, objects $\mathcal{P}$ in $\mathcal{D}$ and $\underline{\mathcal{P}}$ in $\mathcal{C}$ there exists a canonical bijection between all

(i) S^*-couniversal pairs \$$(\underline{\mathcal{P}}, \mathcal{P} v\, I_{\mathcal{P}}\, \underline{\mathcal{P}})$ with a unitary and multiplicative for property S_T^*;

(ii) pairs of natural 2-transformations

$$\varphi^* : \mathcal{C}(\,\cdot\,, \mathcal{P}) \to \mathcal{D}(\,\cdot\, v, \underline{\mathcal{P}}), \quad \psi^* : \mathcal{D}(\,\cdot\, v, \underline{\mathcal{P}}) \to \mathcal{C}(\,\cdot\,, \mathcal{P})$$

of 2-functors from $\mathcal{C}$ to **Cat** such that $\psi^* \circ \varphi^* = 1$.

12. We denote by $\mathcal{C}_u$ the full 2-subcategory of $\mathcal{C}$, for which the objects are all objects $\mathcal{X}$ in $\mathcal{C}$ such that there is a S-universal pair $(\underline{\mathcal{X}}, \mathcal{X}\, J\, (\underline{\mathcal{X}} u))$ from $\mathcal{X}$ to u and $\gamma : \mathcal{C}_u \to \mathcal{C}$ is the canonical embedding.

Fix a S-universal pair $(\underline{\mathcal{X}}, \mathcal{X}\, J_{\mathcal{X}}\, (\underline{\mathcal{X}}\, u))$ from $\mathcal{X}$ to u for any object $\mathcal{X}$ in $\mathcal{C}_u$. Then assigning to each

(i) object $\mathcal{X}$ the object $\underline{\mathcal{X}}$;

(ii) morphism $\mathcal{X}\, K\, \mathcal{Y}$ the morphism $\underline{\mathcal{X}}\, \underline{K}\, \underline{\mathcal{Y}}$, which is uniquely determined by condition $J_{\mathcal{X}} \circ \underline{K} u = K \circ J_{\mathcal{Y}}$ and the property S_M;

(iii) transformation $(\mathcal{X}\, K\, \mathcal{Y})\ \rho\ (\mathcal{X}\, L\, \mathcal{Y})$ the transformation $(\underline{\mathcal{X}}\, \underline{K}\, \underline{\mathcal{Y}})\ \underline{\rho}\ (\underline{\mathcal{X}}\, \underline{L}\, \underline{\mathcal{Y}})$, which is uniquely determined by properties $\rho \cdot e(J_{\mathcal{Y}}) = e(J_{\mathcal{X}}) \cdot \underline{\rho} u$ and S_T;

(iv) object $\mathcal{X}$ the morphism $\mathcal{X}\, J_{\mathcal{X}}\, \mathcal{X}(v \circ u)$ we get a 2-functor $\mathcal{C}_u\, v\, \mathcal{D}$ and a natural 2-transformation from γ to $v \circ u$.

In the dual case we denote by $\mathcal{D}^v$ the full 2-subcategory of $\mathcal{D}$ defined by all objects $\mathcal{P}$ in $\mathcal{D}$ such that there is a S^*-couniversal pair $(\underline{\mathcal{P}}, \underline{\mathcal{P}}\, v)\ I\ \mathcal{P})$ from $\mathcal{C}\, v\, \mathcal{D}$ to $\mathcal{P}$ and by $\delta : \mathcal{D}^v \to \mathcal{D}$ the canonical embedding. Then the 2-functor $\mathcal{D}^v\, u\, \mathcal{C}$ and the natural 2-transformation from $u \circ v$ to δ are similarly determined.

13. Let $\mathcal{C}_u\, v\, \mathcal{D}$ be the 2-functor associated with a system of fixed S-universal pairs from all $\mathcal{X} \in (\mathcal{C}_u)_0$ to u. Suppose that S_T is unitary and multiplicative for composite property. Then for any object $\mathcal{P}$ in $\mathcal{X}$ the systems of functors

$$\varphi : \mathcal{D}(\mathcal{X} v, \mathcal{P}) \to \mathcal{C}(\mathcal{X}, \mathcal{P} u), \quad \psi : \mathcal{C}(\mathcal{X}, \mathcal{P} u), \to \mathcal{D}(\mathcal{X} v, \mathcal{P}) \text{ for all } \mathcal{X} \in \mathcal{C}_0$$

constitute the natural 2-transformations

$$\varphi: \mathcal{D}(\bullet v, \mathcal{P}) \to \mathcal{C}(\bullet, \mathcal{P}u), \quad \psi: \mathcal{C}(\bullet, \mathcal{P}u) \to \mathcal{D}(\bullet v, \mathcal{P})$$

of 2-functors from $\mathcal{C}_u^{op}$ to **Cat**.

Dual assertions are true for the 2-functor $\mathcal{D}^V u\, \mathcal{C}$ associated with a system of fixed S^*-couniversal pairs from $\mathcal{C}\, v\, \mathcal{D}$ to all $\mathcal{P} \in (\mathcal{D}^V)_0$.

14. ***Theorem.*** Given a 2-functor $\mathcal{D}\, u\, \mathcal{C}$ and a full 2-subcategory $\mathcal{E}$ of $\mathcal{C}$ with an embedding functor $\mathcal{E}\, \gamma\, \mathcal{C}$, there exists a canonical bijection between all

(i) pairs (v, J), where $\mathcal{E}\, v\, \mathcal{D}$ is a 2-functor and J: $\gamma \to v \circ u$ is a natural 2-transformation such that for any $\mathcal{X} \in \mathcal{E}_0$, $(\mathcal{X} v, \mathcal{X} J_{\mathcal{X}}\, \mathcal{X}(v \circ u))$ is a S-universal pair from $\mathcal{X}$ to u with unitary and multiplicative for property S_T;

(ii) triples (v, φ, ψ), where $\mathcal{E}\, v\, \mathcal{D}$ is a 2-functor,

$$\varphi: \mathcal{D}(\bullet v, \bullet) \to \mathcal{C}(\bullet, \bullet u), \quad \psi: \mathcal{C}(\bullet, \bullet u) \to \mathcal{D}(\bullet v, \bullet)$$

are binatural 2-transformations of 2-functors from $\mathcal{E}^{op} \times \mathcal{D}$ to **Cat** such that $\psi \circ \varphi = 1$.

A similar canonical bijection exists in the dual case for a 2-functor $\mathcal{C}\, v\, \mathcal{D}$ and a full 2-subcategory $\mathcal{F}$ of $\mathcal{D}$ with an embedding functor $\mathcal{F}\, \delta\, \mathcal{D}$.

15. *Remarks.*

(a) In the case, when all morphisms and transformations of the category $\mathcal{D}$ have the property S, we have both equalities $\psi \circ \varphi = 1$ and $\varphi \circ \psi = 1$. Consequently, in this case φ is a bijection, ψ is the inverse bijection, therefore, ψ is functorial and φ, ψ determine reciprocally inverse isomorphisms of natural 2-transformations. In general case we have not $\varphi \circ \psi = 1$, but only $(\varphi \circ \psi)^2 = \varphi \circ \psi$. The similar remark is correct in S^*-couniversal case.

(b) There is a simple method to reduce the results and notions for 2-categories to 1-categories by interpreting 1-categories as trivial 2-categories, which have only identity transformations.

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