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Mathematics

ON q-BILATTICES

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In this paper the concept of q-bilattice is studied. Interlaced q-bilattices are characterized by the pair of congruencies.

Keywords: q-semilattice, q-lattice, q-bilattice, an interlaced q-bilattice, hyperidentity.

1. Introduction. Bilattices are algebraic structures that were introduced by Ginsberg [1, 2] as a general and uniform framework for a diversity of applications in artificial intelligence. In a series of papers it was shown that these structures may serve as a foundation for many areas, such as logic programming [3–5].

A bilattice is an algebra $(L;\cap,\cup,*,\Delta)$ with four binary operations, for which the following two reducts $L_1=(L;\cap,\cup)$ and $L_2=(L;*,\Delta)$ are lattices.

The bilattice is called interlaced, if all the basic bilattice operations are order preserving with respect to both orders.

In papers [1, 3, 6, 7] bounded distributive or interlaced bilattices were studied. In [8] interlaced bilattices without bounds were characterized (see also [9, 10]).

The algebra $(L;\cap)$ is called a q-semilattice, if it satisfies the following identities:

1. $a\cap b=b\cap a$;
2. $a\cap(b\cap c)=(a\cap b)\cap c$;
3. $a\cap(b\cap b)=a\cap b$.

The algebra $(L;\cap,\cup)$ is called a q-lattice (see [11]), if the reducts $(L;\cap)$ and $(L;\cup)$ are q-semilattices and the following identities $a\cap(b\cup a)=a\cap a$, $a\cup(b\cap a)=a\cup a$, $a\cap a=a\cup a$ are valid.

For each q-semilattice $(L;\cap)$ there is a corresponding quasiorder Q (i.e. a reflexive and transitive relation), defined in the following manner: $aQb\leftrightarrow a\cap b=a\cap a$. For each q-lattice $(L;\cap,\cup)$, we have: $aQb\leftrightarrow a\cap b=$ $=a\cap a\leftrightarrow a\cup b=b\cup b$.

A q-bilattice is an algebraic structure $(L;\cap,\cup,*,\Delta)$ with two q-lattice reducts $L_1=(L;\cap,\cup)$ and $L_2=(L;*,\Delta)$, which also satisfies the following identity

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$a*a=a\cap a$. (The quasiorder of the first reduct $(L;\cap,\cup)$ is denoted by $\leq_\cap$, and the quasiorder of the second reduct by $\leq_*$).

The operation $*$ of the q-semilattice $(L;*)$ is called interlaced with the operations $\cap$ and $\cup$ of the q-lattice $(L;\cap,\cup)$, if the q-semilattice operation $*$ preserves the q-lattice quasiorder, and q-lattice operations $\cap$ and $\cup$ preserve the q-semilattice quasiorder. Note that the operations of a q-lattice are interlaced with each other.

The q-bilattice $(L;\cap,\cup,*,\Delta)$ is called interlaced, if all the basic q-bilattice operations are quasiorder preserving with respect to both quasiorders.

In the present work interlaced q-bilattices are studied.

We need the concept of a hyperidentity and a superproduct of algebras [12, 13].

Let us recall that a hyperidentity is a second-ordered formula of the following type:

$$\forall X_1,\dots,X_m\forall x_1,\dots,x_n(w_1=w_2),$$

where $X_1,\dots,X_m$ are functional variables and $x_1,\dots,x_n$ are objective variables in the words (terms) w_1,w_2. Hyperidentities are usually written without quantifiers: $w_1=w_2$. We say that the hyperidentity $w_1=w_2$ is satisfied in the algebra $(Q;F)$, if this equality is valid, when every objective variable and every functional variable in it is replaced by any element of Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement).

The reader is reffered to [14–16] for characterization of hyperidentities of varieties of lattices, modular lattices, distributive lattices and Boolean algebras. For hyperidentities in thermal (polynomial) algebras (see [17, 18].

For the categorical definition of a hyperidentity in [12] the (bi)homomorphisms between two algebras (Q,F) and $(Q';F')$ are defined as a pair $(\varphi,\tilde{\psi})$ of maps:

$$\varphi:Q\to Q',\tilde{\psi}:F\to F',|A|=|\tilde{\psi}A|,$$

with the following condition

$$\varphi A(a_1,\dots,a_n)=(\tilde{\psi}A)(\varphi a_1,\dots,\varphi a_n)$$

for any $A\in F$, $|A|=n$, $a_1,\dots,a_n\in Q$. For an application of such morphisms in the cryptography see [19].

Algebras and their (bi)homomorphisms $(\varphi,\tilde{\psi})$ (as morphisms) form a category with a product. The product in this category is called a superproduct of algebras and denoted by $Q\bowtie Q'$ for algebras Q and Q'. For example, a superproduct of two q-lattices $Q(+,\cdot)$ and $Q'(+,\cdot)$ is the binary algebra $Q\times Q'((+,+),(\cdot,\cdot),(+,\cdot),(\cdot,+))$ with four binary operations, where the pairs of the operations act component-wise, i.e. $(A,B)((x,y),(u,v))=(A(x,u),B(y,v))$, and $Q\bowtie Q'$ is a q-bilattice. In fact, let us show that $Q\times Q'((+,+),(\cdot,\cdot))$ and $Q\times Q'((+,\cdot),(\cdot,+))$ are q-lattices and satisfy the identity $(+,+)((x,y),(x,y))=$ $=(+,\cdot)((x,y),(x,y))$. The commutativity and associativity are obvious. For other identities we have:

A. $((a,b)(+,+)(c,d))(+,+)(c,d)=((a+c)+c,(b+d)+d)=(a+c,b+d)=$
$=(a,b)(+,+)(c,d)$;
$((a,b)(+,\cdot)(c,d))(+,\cdot)(c,d)=((a+c)+c,(b\cdot d)\cdot d)=(a+c,b\cdot d)=(a,b)(+,\cdot)(c,d)$;
$((a,b)(\cdot,+)(c,d))(\cdot,+)(c,d)=((a\cdot c)\cdot c,(b+d)+d)=(a\cdot c,b+d)=(a,b)(\cdot,+)(c,d)$;
$((a,b)(\cdot,\cdot)(c,d))(\cdot,\cdot)(c,d)=((a\cdot c)\cdot c,(b\cdot d)\cdot d)=(a\cdot c,b\cdot d)=(a,b)(\cdot,\cdot)(c,d)$;

B. $(a,b)(+,+)((c,d)(\cdot,\cdot)(a,b))=(a+(c\cdot a),b+(d\cdot b))=(a+a,b+b)=$
$=(a,b)(+,+)(a,b)$;
$(a,b)(\cdot,\cdot)((c,d)(+,+)(a,b))=(a\cdot(c+a),b\cdot(d+b))=(a\cdot a,b\cdot b)=(a,b)(\cdot,\cdot)(a,b)$;
$(a,b)(+,\cdot)((c,d)(\cdot,+)(a,b))=(a+(c\cdot a),b\cdot(d+b))=(a+a,b\cdot b)=(a,b)(+,\cdot)(a,b)$;
$(a,b)(\cdot,+)((c,d)(\cdot,+)(a,b))=(a\cdot(c+a),b+(d\cdot b))=(a\cdot a,b+b)=(a,b)(\cdot,+)(a,b)$;

C. $(a,b)(+,+)(a,b)=(a+a,b+b)=(a\cdot a,b\cdot b)=(a,b)(\cdot,\cdot)(a,b)$;
$(a,b)(+,\cdot)(a,b)=(a+a,b\cdot b)=(a\cdot a,b+b)=(a,b)(\cdot,+)(a,b)$;

D. $(a,b)(+,+)(a,b)=(a+a,b+b)=(a+a,b\cdot b)=(a,b)(+,\cdot)(a,b)$.

It is easy to show that quasiorders on the q-lattices $Q\times Q'((+,+),(\cdot,\cdot))$ and $Q\times Q'((+,\cdot),(\cdot,+))$, which are denoted by the symbols $\le_{\mathrm{I}}$ and $\le_{\mathrm{II}}$ correspondingly, are defined by the following rules:

$$(a,b)\le_{\mathrm{I}}(c,d)\leftrightarrow a\le_1 c \text{ and } b\le_2 d;$$
$$(a,b)\le_{\mathrm{II}}(c,d)\leftrightarrow a\le_1 c \text{ and } d\le_2 b,$$

where $\le_1$ and $\le_2$ are the quasiorders on the q-lattices $Q(+,\cdot)$ and $Q'(+,\cdot)$. So,

$$(a,b)\le_{\mathrm{I}}(c,d)\,\&\,(e,f)\le_{\mathrm{I}}(g,h)\to a\le_1 c\,\&\,b\le_2 d\,\&\,e\le_1 g\,\&\,f\le_2 h\to$$
$$\to a+e\le_1 c+g,\ a\cdot e\le_1 c\cdot g\,\&\,b+f\le_2 d+h,\ b\cdot f\le_2 d\cdot h\to$$
$$\to(a+e,b+f)\le_{\mathrm{I}}(c+g,d+h)\,\&\,(a\cdot e,b\cdot f)\le_{\mathrm{I}}(c\cdot g,d\cdot h);$$
$$(a,b)\le_{\mathrm{II}}(c,d)\,\&\,(e,f)\le_{\mathrm{II}}(g,h)\to a\le_1 c\,\&\,d\le_2 b\,\&\,e\le_1 g\,\&\,h\le_2 f\to$$
$$\to a+e\le_1 c+g,\ a\cdot e\le_1 c\cdot g\,\&\,d+h\le_2 b+f,\ d\cdot h\le_2 b\cdot f\to$$
$$\to(a+e,b+f)\le_{\mathrm{II}}(c+g,d+h)\,\&\,(a\cdot e,b\cdot f)\le_{\mathrm{II}}(c\cdot g,d\cdot h).$$

Hence, $Q\times Q'((+,+),(\cdot,\cdot),(+,\cdot),(\cdot,+))$ is an interlaced q-bilattice.

2. Some Lemmas.

2.1. Congruence relations Θ,Φ of a q-lattice $(L;\cap,\cup)$ satisfying to the following conditions: $a\Theta a\cap a$ and $a\Phi a\cap a$, commute iff for each $a,b\in L$, $a\le b\to a\Theta\Phi b$ is equivalent to $a\Phi\Theta b$

Proof. The condition is obviously necessary. Let's show that it is sufficient too. Suppose $x,y\in L$, $x\Theta z$ and $z\Phi y$, hence, $x\cap x\Theta z$, $z\Phi y\cap y$. Then $x\cap y\cap z\Phi x\cap z\Theta x\cap x$, and it follows that there exists $t\in L$ such that $x\cap y\cap z\Theta t\Phi x\cap x$, so $y\cup y\Theta y\cup t$. Further, $x\cap y\cap z\Theta y\cap z\Phi y\cap y$, then $y\cap z\Phi\Theta y\cup t$, $y\cap z\Theta\Phi y\cup t$ and $t\Theta y\cap z$, so $t\Theta\Phi y\cup t$, $x\cap x\Phi t\Phi\Theta y\cup t\Theta y\cup y$, hence, $x\cap x\Phi\Theta y\cap y$. This shows that $\Theta\Phi\le\Phi\Theta$ so $\Theta\Phi=\Phi\Theta$. □

2.2. The operation $*$ of a q-semilattice $(L;*)$ is interlaced with the operations $\cap$ and $\cup$ of the q-lattice $(L;\cap,\cup)$, iff the following hyperidentity is satisfied in the algebra $(L;\cap,\cup,*)$:

$$X(Y(X(x,y),z),Y(y,z))=X(Y(X(x,y),z),\,Y(X(x,y),z)).$$

Proof. Let us show, for example, that $[(x\cap y)*z]\cap(y*z)=$ $=[(x\cap y)*z]\cap[(x\cap y)*z]$ follows from $x\le_\cap y\to x*z\le_\cap y*z$ and conversely.

$(\to)$ $x\cap y\le_\cap y$ for any $x,y\in L$. Then $(x\cap y)*z\le_\cap y*z$. So, $[(x\cap y)*z]\cap(y*z)=[(x\cap y)*z]\cap[(x\cap y)*z]$.

$(\leftarrow)$ $x*z\le_\cap(x\cap x)*z\le_\cap x*z$ for any $x,y,z\in L$, then $[(x\cap x)*z]\cap[(x\cap x)*z]=$ $=(x*z)\cap(x*z)$. Let $x\le_\cap y$, then $x\cap y=x\cap x$. In that case $(x*z)\cap(y*z)=$ $=(x*z)\cap(x*z)\cap(y*z)=[(x\cap x)*z]\cap[(x\cap x)*z]\cap(y*z)=[(x\cap y)*z]\cap(y*z)=$ $=[(x\cap y)*z]\cap[(x\cap y)*z]=[(x\cap x)*z]\cap[(x\cap x)*z]=(x*z)\cap(x*z)$. Hence, $x*z\le_\cap y*z$. □

2.3. The operation $*$ of a q-semilattice $(L;*)$ is interlaced with the operations $\cap$ and $\cup$ of the q-lattice $(L;\cap,\cup)$, for which $x*x=x\cap x$, iff the algebra $(L;\cap,\cup,*)$ satisfies the following hyperidentity:

$$X(Y(X(x,y),z),Y(y,z))=Y(X(x,y),z). \quad □$$

In the propositions 2.4–2.17 we suppose that $(L;\cap,\cup)$ is a q-lattice, $(L;*)$ is a q-semilattice and the operation $*$ is interlaced with the operations $\cap,\cup$ and satisfies the identity $a\cap a=a*a$.

2.4. $x\cap y\le_\cap x*y\le_\cap x\cup y,\ x*y\le_* x\cap y,\ x*y\le_* x\cup y$.

2.5. $X(Y(x,y),Y(x,y))=Y(x,y)$, where $X,Y\in\{\cap,\cup,*\}$ for any $x,y\in L$.

2.6. $a\le_\cap x\le_\cap b\ \&\ a\le_* b\to a\le_* x\le_* b$; $a\le_\cap x\le_\cap b\ \&\ b\le_* a\to b\le_* x\le_* a$.

Proof. If we suppose $a\le_* b$, then $a\cap x\le_* b\cap x$ and $a\cup x\le_* b\cup x$. Since, $a\le_\cap x\le_\cap b$, then $a\cap a\le_* x\cap x$ and $x\cap x\le_* b\cap b$, hence, $a\le_* x\le_* b$. The second statement can be proved analogously. □

2.7. $u\le_\cap x\ \&\ u\le_\cap y\ \&\ u\le_* x\ \&\ u\le_* y\to x\cap y=x*y$;

$x\le_\cap u\ \&\ y\le_\cap u\ \&\ u\le_* x\ \&\ u\le_* y\to x\cup y=x*y$.

Proof. We have $u\cap u\le_\cap x\cap y\le_* x*y$, $u\le_* x$ and $u\le_* y$, hence, $u\cap u=u*u\le_* x*y$, then $x\cap y\le_* x*y$. Similarly, $x*y\le_* x\cap y$, so, $x*y\le_* x\cap y\le_* x*y$, hence, $x\cap y=x*y$. □

2.8. Let the q-semilattice $(L;*)$ forms a q-lattice $(L;*,\Delta)$. Then

$$a\le_\cap b\to a\le_\cap a\Delta b\le_\cap b.$$

Proof. By 2.4, $a\cap a=a*a=a*(a\Delta b)\le_* a\cup(a\Delta b)$, then from $a\le_\cap b$ we get $a*a=a\cap a=b\cap a=b\cap(a\cap a)\le_* b\cap[a\cup(a\Delta b)]$, hence,

$$a*a\le_* b\cap[a\cup(a\Delta b)]. \quad (1)$$

From $b\le_* a\Delta b$ and $a\le_\cap b$ we obtain $b\cap b=a\cup b\le_* a\cup(a\Delta b)$, hence, $b*b=b\cap b=(b\cap b)\cap b\le_* b\cap[a\cup(a\Delta b)]$, so,

$$b*b\le_* b\cap[a\cup(a\Delta b)]. \quad (2)$$

From (1) and (2) it follows that

$$(a*a)\Delta(b*b)\le_*(b\cap[a\cup((a\Delta b))])\Delta(b\cap[a\cup(a\Delta b)])=$$
$$=(b\cap[a\cup(a\Delta b)])\cap(b\cap[a\cup(a\Delta b)])=b\cap[a\cup(a\Delta b)].$$

Then $a\Delta b\le_* b\cap[a\cup(a\Delta b)]$.

Further, from $a \le_* a\Delta b$ it follows that

$$a \cup (a\Delta b) \le_* (a\Delta b) \cup (a\Delta b) = a\Delta b,\ a \cap (a\Delta b) \le_* (a\Delta b) \cap (a\Delta b) = a\Delta b.$$

From $b \le_* a\Delta b$ we deduce that $b \cup (a\Delta b) \le_* a\Delta b,\ b \cap (a\Delta b) \le_* a\Delta b.$

So, $a\Delta b \le_* b \cap [a \cup (a\Delta b)] \le_* b \cap (a\Delta b) \le_* a\Delta b,$ hence,

$$[b \cap (a\Delta b)] * [b \cap (a\Delta b)] = (a\Delta b) * (a\Delta b).$$

$$[b \cap (a\Delta b)] * [b \cap (a\Delta b)] = [b \cap (a\Delta b)] \cap [b \cap (a\Delta b)] = b \cap (a\Delta b),$$

$$(a\Delta b) * (a\Delta b) = (a\Delta b) \cap (a\Delta b)\text{, hence, } b \cap (a\Delta b) = (a\Delta b) \cap (a\Delta b).$$

So, $a\Delta b \le_\cap b$. The second part of the inequality can be proved the same way. □

Define the relations θ_1 and θ_2 in $(L;\cap,\cup,*)$ as follows:

$$a\theta_1 b \leftrightarrow a * b = a \cup b;\quad a\theta_2 b \leftrightarrow a * b = a \cap b.$$

2.9. θ_2 is an equivalence relation in $(L;\cap,\cup)$.

Proof. Reflexivity and symmetry are clear. Let $a\theta_2 b$ and $b\theta_2 c$, then $a * b = a \cap b$ and $b * c = b \cap c$. Hence, $a \cap b \le_* b$ and $b \cap c \le_* b$. Then $a \cap b \cap c \le_* b \cap c$ and $a \cap b \cap c \le_* a \cap b$, hence, $(a \cap b \cap c) \cap (a \cap b \cap c) \le_* (a \cap b) * (b \cap c) = (a * b) * (b * c) = a * (b * b) * c = a * b * c$. On the other hand, $a * b * c \le_* a \cap b \cap c$. So, $a \cap b \cap c = a * b * c$, hence, $a \cap b \cap c \le_* a,\ a \cap b \cap c \le_* c,\ a \cap b \cap c \le_\cap a,$ and $a \cap b \cap c \le_\cap c.$ Then, by 2.7, $a \cap c = a * c$, which shows that gives $a\theta_2 c$. □

2.10. θ_2 is a congruence relation in $(L;\cap,\cup)$.

Proof. Let $a\theta_2 b$, hence, $a * b = a \cap b$. So, $a \cap b \le_* a$ and $a \cap b \le_* b$. Then for any $c \in L$ it follows that $a \cap b \cap c \le_* a \cap c$ and $a \cap b \cap c \le_* b \cap c$ and since $a \cap b \cap c \le_\cap a \cap c$ and $a \cap b \cap c \le_\cap b \cap c$, we get by 2.7 that $(a \cap c) \cap (b \cap c) = (a \cap b) * (b \cap c)$, hence, $a \cap c\theta_2 b \cap c$. Similarly we get that $a \cup c\theta_2 b \cup c$. □

2.11. θ_1 is a congruence relation in $(L;\cap,\cup)$.

2.12. θ_1 and θ_2 are congruence relations in $(L;*)$.

Proof. Let $a\theta_1 b$, i.e. $a * b = a \cup b$, then $a \le_\cap a * b$ and $b \le_\cap a * b$. So, $a * c \le_\cap a * b * c$ and $b * c \le_\cap a * b * c$. Since, $a * b * c \le_* a * c$ and $a * b * c \le_* b * c$, then from 2.7 we obtain $(a * c) * (b * c) = (a * c) \cup (b * c)$, hence $a * c\theta_1 b * c$.

In the same way we can show that $a * c\theta_2 b * c$ follows from $a\theta_2 b$. □

2.13. $a(\theta_1 \cap \theta_2)b \leftrightarrow a \cap a = b \cap b$.

Proof. $a(\theta_1 \cap \theta_2)b \leftrightarrow a\theta_1 b$ and $a\theta_2 b \leftrightarrow a * b = a \cup b$ and $a * b = a \cap b \leftrightarrow a \cup b = a \cap b \leftrightarrow a \cap a = b \cap b$. □

2.14. $a \cap b\theta_1 a * b,\ a * b\theta_2 a \cup b$.

Proof. $(a \cap b) * (a * b) = a * b = (a \cap b) \cup (a * b)$, hence, $a \cap b\theta_1 a * b$. By 2.4, we have $(a \cup b) * (a * b) = a * b = (a \cup b) \cap (a * b)$, hence, $a \cup b\theta_2 a * b$. □

2.15. $a \le_\cap b \rightarrow a\theta_1\theta_2 b$.

Proof. By 2.14, $a \cap b\theta_1 a * b$ and $a * b\theta_2 a \cup b$, giving us $a \cap b\theta_1\theta_2 a \cup b$, hence, $a \cap a\theta_1\theta_2 b \cap b$, so $a\theta_1\theta_2 b$.

2.16. $a \leq_{\cap} b \rightarrow a\theta_2\theta_1 b$. □

Proof. Using 2.8 we get $a\theta_2 a \Delta b\theta_1 b$. □

2.17. L/θ_1 and L/θ_2 are lattices.

Proof. Note that $a\theta_i a \cap a$ and $a\theta_i a \cup a$, for $i = 1,2$. Hence, the elements of quantient-algebras, L/θ_1 and L/θ_2, are idempotent. □

3. Theorems.

Theorem 1. Let $(L;\cap,\cup)$ be a q-lattice, $(L;*)$ be a q-semilattice, the operation of which is interlaced with the operations $\cup$ and $\cap$, and satisfy the identity $a \cap a = a * a$. Then there exists a pair of congruences (θ_1, θ_2) in the q-lattice $(L;\cap,\cup)$, satisfying the following conditions:

1. $a(\theta_1 \cap \theta_2)b \leftrightarrow a \cap a = b \cap b$;
2. $a \leq_{\cap} b \rightarrow a\theta_1\theta_2 b$;
3. $X(Y(X(x,y),z),Y(y,z))\theta_i Y(X(x,y),z)$ for $i = 1,2$, (3)

where $X, Y \in \{\cap,\cup\}$, $x, y, z \in L$.

Conversely, each pair of congruences (θ_1,θ_2) in $(L;\cap,\cup)$ satisfying the conditions 1–3, corresponds to a q-semilattice $(L;*)$, the operation of which is interlaced with the operations $\cup$ and $\cap$ and satisfies the identity $a \cap a = a * a$.

Proof. Define the relations θ_1 and θ_2 as above. From 2.10, 2.11, 2.13 and 2.15 we get that θ_1, θ_2 are congruences in $(L;\cap,\cup)$ satisfying conditions 1 and 2. The condition 3 is valid, since any q-lattice is interlaced.

Conversely, let θ_1 and θ_2 are congruences satisfying the conditions of Theorem 1. Define the operation $*$ by the following rule:

$$a * b = d \cap d \leftrightarrow d\theta_1 a \cap b \text{ and } d\theta_2 a \cup b.$$

The existence of such $d \in L$ follows from the Condition 2. Obviously, the operation $*$ is commutative, and the following identities are true: $a*(b*b) = a*b$, $a \cap a = a * a$. The elements $d_1 = (a*b)*c$, $d_2 = a*(b*c)$ satisfy $d_1 \cap d_1\theta_1 a \cap b \cap c$ and $d_1 \cap d_1\theta_2 a \cup b \cup c$, $d_2 \cap d_2\theta_1 a \cap b \cap c$ and $d_2 \cap d_2\theta_1 a \cup b \cup c$ consequently, $d_1\theta_1 d_2$ and $d_1\theta_2 d_2$, hence, $d_1 \cap d_1 = d_2 \cap d_2$, so by 2.5, $(a*b)*c = a*(b*c)$.

To prove that the operation $*$ is interlaced with the operations $\cap,\cup$, we use the definition of the operation $*$ and the fact that the q-lattices L/θ_i are interlaced $(i = 1,2)$. For example, the elements $u_1 = (x*y) \cap z$ and $u_2 = [(x*y) \cap z]*(y \cap z)$ satisfy $x \cap y \cap z\theta_1 u_1\theta_2 (x \cup y) \cap z$ and $x \cap y \cap z\theta_1 u_2\theta_2 (x \cup y) \cap z$, then $u_1 \cap u_1 = u_2 \cap u_2 \rightarrow (x*y) \cap z = [(x*y) \cap z]*(y \cap z)$ (by 2.5). □

Theorem 2. Let $(L;\cap,\cup)$ be a q-lattice. There exists a bijective correspondence between the q-semilattice operations $*$ in L, which are interlaced with the operations $\cap, \cup$ and satisfy the identity $a \cap a = a * a$, and the epimorphism φ acting from $(L;\cap,\cup)$ to the subdirect product of two lattices, satisfying $\varphi(x) = \varphi(y) \leftrightarrow x \cap x = y \cap y$. Moreover, if $(a,b),(a',b')$ are elements of this subdirect product and $(a,b) \leq_{\cap} (a',b')$, then (a,b') belongs to this subdirect product too, and if $\varphi(x)=(a,b)$, $\varphi(y)=(a',b')$, then $\varphi(x*y)=(a\cap a', b\cup b')$.

Proof. Let $(L;*)$ be a q-semilattice satisfying the theorem's conditions, and θ_1 and θ_2 are the congruence relations from Theorem 1. Then the q-lattice $(L;\cap,\cup)$ is epimorphically mapped to the subdirect product of the two lattices L/θ_1 and L/θ_2, $\varphi: x \to ([x]_{\theta_1},[x]_{\theta_2})$, such that $\varphi(x)=\varphi(y) \leftrightarrow x\cap x = y\cap y$.

Let (a,b) and (a',b') belong to this subdirect product and $(a,b)\leq_{\cap}(a',b')$. Then there exist $u,v\in L$ such that $\varphi(u)=(a,b), \varphi(v)=(a',b')$ and $u\leq_{\cap} v$. Then by the Theorem 1there exists $t\in L$ with the property $u\theta_1 t\theta_2 v$, hence, $\varphi(t)=(a,b')$.

Conversely, let φ be an epimorphism between L and the subdirect product of lattices A and B satisfying the Theorem's conditions. Then there exist congruence relations θ_1 and θ_2 such that $a(\theta_1\cap\theta_2)b \leftrightarrow a\cap a = b\cap b$, and L/θ_1, L/θ_2 are isomorphic to A and B respectively. So, L/θ_i satisfy the condition (1).

If $a,b\in L$ and $a\leq_{\cap} b$, then $\varphi(a)\leq_{\cap}\varphi(b)$, so, by our assumption we get that $([a]_{\theta_1},[b]_{\theta_2})$ belongs to the subdirect product, hence there exists $t\in L$ such that $\varphi(t)=([a]_{\theta_1},[b]_{\theta_2}) \to a\theta_1 t$ and $t\theta_2 b \to a\theta_1\theta_2 b$. Hence, the pair of congruences (θ_1,θ_2) in L satisfies the conditions of Theorem 1, and it follows that there exists a q-semilattice operation, $*$, which is interlaced with the operations $\cap$ and $\cup$, and satisfies the identity $a\cap a = a*a$. The last statement of the theorem can be proved with the help of the relation $a\cap b\theta_1 a*b\theta_2 a\cup b$. □

Theorem 3. a) The q-semilattice $(L;*)$ from Theorem 1 can be transformed into a lattice $(L;*,\Delta)$ (moreover, in the unique way), iff the corresponding congruences θ_1 and θ_2 commute.

b) The q-semilattice $(L;*)$ of Theorem 2 can be transformed into a q-lattice $(L;*,\Delta)$, iff the subdirect product turns out to be a direct.

Proof. a) If the lattice $(L;*,\Delta)$ exists, then $\theta_1\theta_2=\theta_2\theta_1$ by 2.1, 2.15 and 2.16. Conversely, let $\theta_1\theta_2=\theta_2\theta_1$. Then for $a\leq_{\cap} b$ we get by Theorem 1 $a\theta_1\theta_2 b$, hence, $a\theta_2\theta_1 b$, too. By Theorem 1 there exists a q-semilattice operation Δ in L corresponding to the pair (θ_2,θ_1), which is interlaced with the operations $\cap$ and $\cup$, and satisfies the identities $a\Delta a = a\cap a$, $a\cap b\theta_2(a\Delta b)\theta_1 a\cup b$. Hence, $(a\Delta b)*a\theta_2(a\cap b)*a\theta_2(a\cap b)\cup a = a\cup a$ and $(a\Delta b)*a\theta_1(a\cup b)*a\theta_1(a\cup b)\cap a =$ $= a\cap a \to (a\Delta b)*a(\theta_1\cap\theta_2)a\cap a \to a*a=(a\Delta b)*a$. Similarly, we get $(a*b)\Delta a = a\Delta a$. Hence, $(L;*,\Delta)$ is a q-lattice.

b) Let $a,b\in L$ then by 2.14 and 2.4 $a\cap b\theta_1 a*b\theta_2 a\cup b$ and $a\cap b\leq_{\cap} a*$ $*b\leq_{\cap} a\cup b$, then $\theta_1\cup\theta_2=i$ [20]. So, the subdirect product is a direct product, iff $\theta_2\theta_1=\theta_1\theta_2$ [20]. By a), this is equivalent to the condition that $(L;*)$ forms a q-lattice. □

Theorem 4. Let $(L;\cap,\cup)$ and $(L;*,\Delta)$ be q-lattices. If the operation $*$ is interlaced with the operations $\cap$ and $\cup$, and satisfies the identity $a\cap a = a*a$

then the operation Δ is interlaced with the operations $\cap$ and $\cup$ too.

Proof. The proof follows from Theorems 1 and 3. □

Theorem 5. Let $L_1 = (L;\cap,\cup)$ and $L_2 = (L;*,\Delta)$ be q-lattices with the following identity: $x \cap x = x * x$. The operation $*$ is interlaced with the operations $\cap,\cup$, iff there exists an epimorphism φ from the q-bilattice $(L;\cap,\cup,*,\Delta)$ to the superproduct of the two lattices $L_1 \bowtie L_2$, such that the epimorphism φ satisfies the following condition: $\varphi(x) = \varphi(y) \leftrightarrow x \cap x = y \cap y$. Hence, the epimorphism φ is an isomorphism on the bilattice of the idempotent elements of the q-bilattice.

Proof. By the Theorems 2 and 3b), there exists an epimorphism $\varphi : L \to A \times B$ between the q-lattice $(L;\cap,\cup)$ and the subdirect product of two lattices A and B, which satisfying the condition $\varphi(x) = \varphi(y) \leftrightarrow x \cap x = y \cap y$. The map φ can be continued to the epimorphism between the q-bilattice $(L;\cap,\cup,*,\Delta)$ and the superproduct $A \bowtie B$ in the following manner:

$$\varphi(x * y) = (a \cap a', b \cup b'); \qquad \varphi(x \Delta y) = (a \cup a', b \cap b'),$$

where $\varphi(x) = (a,b),\ \varphi(y) = (a',b')$. □

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