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Mathematics

EXPLICIT FACTORIZATION OF A (P,Q)-CIRCULANT MATRIX-FUNCTIONS

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The paper considers a factorization problem of a matrix-function, obtained from a circulant by a right and left multiplication by diagonal rational matrix-functions. Formulas for partial indices are obtained by means of ranks of a finite number of explicit type matrices. A factorization construction of this matrix-function based on factorization of finite number of functions is given as well.

Keywords: matrix-function, factorization, partial indices.

1. Let Γ be a Carleson contour, which bounds finitely connected bounded domain $\Omega_+ (0 \in \Omega_+), \Omega_- = \mathbb{C} \setminus \overline{\Omega}_+ (\infty \in \Omega_-)$. It is known (see [1]) that the singular integral operator S, defined by the formula

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{1}{\tau - t} \varphi(\tau) d\tau, \ \ t \in \Gamma,$$

where the integral is understood in a sense of a principal value, is a bounded operator in the space $L_p (= L_p(\Gamma)),\ 1 < p < \infty$. We define projectors $P_\pm = \frac{1}{2}(I \pm S)$ and classes of functions $L_p^+ = \operatorname{Im} P_+$, $\overset{0}{L_p^-} = \operatorname{Im} P_-$, $L_p^- = \overset{0}{L_p^-} + C.$

Everywhere below we denote the space of n-dimensional vector-columns ($n \times n$-order matrices) with elements from the linear space X by X^n ($X^{n\times n}$). The abbreviations m.-f. and v.-f. will be used for matrix-function and vector-function respectively. By $\tau_k (k \in \mathbb{Z})$ we denote a function defined by $\tau_k(t) = t^k$.

By factorization of a m.-f. G of order $n \times n$ in the space L_p along the contour Γ we mean the representation $G = G_- \Lambda G_+^{-1}$, where a) $G_\pm \in (L_p^\pm)^{n\times n}$, $G_\pm^{-1} \in (L_q^\pm)^{n\times n}$, $q = \frac{p}{p-1}$; b) $\Lambda = \operatorname{diag}[\tau_{\kappa_1}, \ldots, \tau_{\kappa_n}]$, where $\kappa_1 \le \ldots \le \kappa_n$ are numbers called partial indices. A factorization of m.-f. G satisfying to the condition $G^{\pm 1} \in L_\infty^{nxn}$ is called generalized, if the operator $G_- P_+ G_-^{-1} I$ is bounded in L_p^n.

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Suppose that $\varphi_i \in L_\infty$, and $p_i, q_i\ (i = 1,...,n)$ are rational functions with poles lying outside the contour Γ, $P = \mathrm{diag}[p_1,\dots,p_n]$, $Q = \mathrm{diag}[q_1,\dots,q_n]$ and $\Phi = (\varphi_{ij})_{i,j=1}^n$, where $\varphi_{ij} = \varphi_{j-i+1}, i \le j$, and $\varphi_{ij} = \varphi_{n+j-i+1}$, $i > j$. The m.-f. Φ is a circulant and, therefore, a m.-f. G, defined by the equality $G = P\Phi Q$, we will call (P,Q)-circulant. The explicit representation of this m.-f. will be:

$$G = \begin{pmatrix} p_1 q_1 \varphi_1 & p_1 q_2 \varphi_2 & \cdots & p_1 q_n \varphi_n \\ p_2 q_1 \varphi_n & p_2 q_2 \varphi_1 & \cdots & p_2 q_n \varphi_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ p_n q_1 \varphi_2 & p_n q_2 \varphi_3 & \cdots & p_n q_n \varphi_1 \end{pmatrix}.$$

The present paper suggests a method of an explicit factorization of the m.-f. G. By the explicit factorization we mean a reduction of a factorization problem of the m.-f. G to a finite number of factorizations of scalar functions and a finite number of solutions of linear algebraic problems. The suggested approach is based on algebraic properties of the Toeplitz operators' family (see [2, 3]) and extends the method developed in [5, 6].

2. We introduce functions $\psi_j = \sum_{s=1}^{n} \varepsilon^{(j-1)(s-1)} \varphi_s$, $j = 1,...,n$, where $\varepsilon = \exp\dfrac{2\pi i}{n}$ and an m.-f. $\Psi = \mathrm{diag}[\psi_1,\dots,\psi_n]$. It is known that $\Phi = E\Psi E^{-1}$, where $E = \left(\varepsilon^{(k-1)(s-1)}\right)_{k,s=1}^n$, $E^{-1} = \dfrac{1}{n}\left(\varepsilon^{-(k-1)(s-1)}\right)_{k,s=1}^n$. Consequently, $G = PE\Psi E^{-1} Q$. By the Theorem 3.10 from [4], we have that the m.-f. G admits a generalized factorization, iff the functions $\psi_j\,(j = 1,..,n)$ admit generalized factorization.

Below, without loss of generality, we will assume that $q_1(0),...,q_n(0) \in \mathbb{C} \setminus \{0\}$.

Let the functions $\psi_j (j = 1,...,n)$ admit the generalized factorization $\psi_j = \psi_j^- t^{\chi_j} (\psi_j^+)^{-1}$. Then we have $G = PE\Psi^- \Lambda_\chi (\Psi^+)^{-1} E^{-1} Q$, where $\Psi^- = \mathrm{diag}[\psi_1^-,...,\psi_n^-]$, $\Lambda_\chi = \mathrm{diag}[\tau_{\chi_1},...,\tau_{\chi_n}]$ and $\Psi^+ = \mathrm{diag}[\psi_1^+,...,\psi_n^+]$.

We define m.-f. $\tilde{A} = \tau_{-\chi_{\max}} PE\Psi\Lambda_\chi$ and $\tilde{B} = (\Psi^+)^{-1} E^{-1} Q$, where $\chi_{\max} = \max\{\chi_1,\dots,\chi_n\}$. Then $G = \tau_{\chi_{\max}} \tilde{A}\tilde{B}$.

Let $p_i = \dfrac{p_{i1}}{p_{i2}}$, $q_i = \dfrac{q_{i1}}{q_{i2}}$. We write the polynomials p_{i1}, p_{i2}, q_{i1}, q_{i2} as follows: $p_{i1} = p_{i1}^- p_{i1}^+$, $p_{i2} = p_{i2}^- p_{i2}^+$, $q_{i1} = q_{i1}^- q_{i1}^+$, $q_{i2} = q_{i2}^- q_{i2}^+$, where $p_{ik}^\pm$, $q_{ik}^\pm$ $(k = 1,2)$ are polynomials, whose zeros lie in $\Omega_\mp$ respectively. We denote by $p_{ok}^\pm (q_{ok}^\pm)\ (k = 1,2)$ polynomials, which are the least common multiples of $p_{1k}^\pm,\dots,p_{nk}^\pm\ (q_{1k}^\pm,\dots,q_{nk}^\pm)$. Let $A = p_{02}^+ \cdot \tau_{-\upsilon_0} \cdot \dfrac{1}{q_{02}^-} \tilde{A}$, $B = \dfrac{q_{02}^-}{p_{02}^+} \tilde{B}$, where

$\upsilon_0 = \max_{i=1,\dots,n} \deg p_{i1} + \deg p_{02}^+$. Then $G = \tau_{\chi_0} AB$, where $A \in (L_p^-)^{nxn}$, $B \in (L_q^+)^{nxn}$, $A^{-1} \in (M_q^-)^{nxn}$, $B^{-1} \in (M_p^+)^{nxn}$, $\chi_0 = \chi_{\max} + \upsilon_0$. We define a number $\upsilon_- = \upsilon_0 + \chi_0 - \chi_{\min} + \deg q_{02}^- + \deg p_{01}^+ + \max_{i=1,\dots,n} \deg p_{i2}$, where $\chi_{\min} = \min\{\chi_1, \dots, \chi_n\}$, and polynomial $q_- = q_{01}^- \cdot q_{02}^-$ (we denote its degree by υ_+). Further, we define a v.-f. $q_+ = p_{01}^+ p_{02}^+$ and $n_- = \deg q_+$. Consider families of Hankel and Toeplitz operators $H_j^- : D_p^-(A^{-1}) \to \overset{0}{L_p^+}, H_j^+ : D_p^+(B^{-1}) \to \overset{0}{L_p^-}, T_j : L_p^+ \to L_p^+$, $p > 1$, defined by formulas $H_j^- \varphi = P_+(\tau_{j^+} A^{-1} \varphi)$, $H_j^+ \varphi = P_-(\tau_{j^-} B^{-1} \varphi), T_j \varphi = P_+(\tau_{-j} AB\varphi)$, where $j^{\pm} = \dfrac{j \pm |j|}{2}$, $D_p^-(A^{-1}) = \{\varphi \in (\overset{0}{L_p^-})^n,\ A^{-1}\varphi \in (L_p)^n\}$; $D_p^+(B^{-1}) = \{\varphi \in (L_p^+)^n,\ B^{-1}\varphi \in (L_p)^n\}$.

We denote by $\Im_j$ the space of vector polynomials $\sum_{k=0}^{j-1} \varphi_k z^k$, $\varphi_k \in \mathbb{C}^n$, in the case when $j > 0$ $(j \in \mathbb{Z})$, and the space of vector polynomials in z^{-1} of type $\sum_{k=j}^{-1} \varphi_k z^k$ in the case when $j < 0$ $(j \in \mathbb{Z})$. We will suppose that $\Im_0 = \{0\}$. We define a family of finite-dimensional operators $K_j = H_j^+ H_j^- \Big|_{\Im_{-(\upsilon_- + j^+)}}$, $j \in \mathbb{Z}$. Denote $N_j = \ker T_j$.

Lemma 1. A v.-f. φ belongs to N_j, iff there exists a v.-f. $\psi \in \ker K_j$, such that $\varphi = \tau_j B^{-1} H_j^-(\psi)$. Besides, the following equality is true:

$$\dim N_j = \upsilon_- + nj^+ - \dim \operatorname{Im} K_j,\ j \in \mathbb{Z}. \tag{1}$$

Proof. It is known that (see [7]) $\operatorname{Im} H_j^- = \operatorname{Im} H_j^- |_{\Im_{-(\upsilon_- + j^+)}}, j \in \mathbb{Z}$. Hence, we have the equality $H_j^-(\ker K_j) = \ker\left(H_j^+ |_{\operatorname{Im} H_j^-}\right)$. Therefore, to prove the first part of Lemma, it is enough to see that $\varphi \in N_j$, iff the v.-f. $\varphi_0 = \tau_{-j^-} B\varphi \in \ker\left(H_j^+ |_{\operatorname{Im} H_j^-}\right)$.

Assume that $\varphi \in N_j$, then $\varphi \in (L_p^+)^n$ and $P_+(\tau_{-j} AB\varphi) = 0$, i.e. $\tau_{-j} AB\varphi = f \in (\overset{0}{L_p^-})^n$. We write the last equality in the form $\tau_{j^+} A^{-1} f = \tau_{-j^-} B\varphi = \varphi_0$.

We have $B \in (L_q^+)^{nxn}$ and $\varphi \in (L_p^+)^n$, hence, $\tau_{j^+} A^{-1} f \in (L_1^+)^n$. Since $A^{-1} \in (M_q^-)^{nxn}$, $f \in (\overset{0}{L_p^-})^n$, then $P_+(\tau_{j^+} A^{-1} f)$ is a rational function, and, therefore, $\tau_{j^+} A^{-1} f \in (L_p^n \bigcap (L_1^+)^n)$, i.e. $f \in \overset{0}{D_p^-}(A^{-1})$. It is easy to see that $H_j^- f = \varphi_0$ and $B^{-1}\varphi_0 = \tau_{-j^-}\varphi \in (L_p)^n$, i.e. $\varphi_0 \in D_p^+(B^{-1})$ and $H_j^+ \varphi_0 = P_-(\tau_{j^-} B^{-1} \varphi_0) = P_-(\varphi) = 0$. Thus, $\varphi_0 \in \ker\left(H_j^+ |_{\operatorname{Im} H_j^-}\right)$.

Conversely, assume that $\varphi_0 = \tau_{-j^-} B\varphi \in \ker\,(H_j^+|_{\mathrm{Im} H_j^-})$. The equality $\mathrm{Im} H_j^- = \mathrm{Im} H_j^-|_{\Im_{-(\upsilon_- + j^+)}}$ implies the existence of $f \in \Im_{-(\upsilon_- + j^+)}$, such that $\varphi_0 = H_j^- f \in (L_p^+)^n$. Since $H_j^+\varphi_0 = 0$, then $0 = P_-(\tau_{j^-} B^{-1}\varphi_0) = P_-(\tau_{j^-} B^{-1}\tau_{-j^-} B\varphi)$ $= P_-(\varphi)$ i.e. $\varphi \in (L_p^+)^n$.

According to the definition of φ_0, $\tau_{j^-} B\varphi = H_j^- f = P_+(\tau_{j^+} A^{-1} f) =$ $= \tau_{j^+} A^{-1} f - P_-(\tau_{j^+} A^{-1} f)$. Consequently, $f - \tau_{j^+} AP_-(\tau_{j^+} A^{-1} f) = \tau_{-j} AB\varphi$. Taking into account that $f \in \Im_{-(\upsilon_- + j^+)}$ and $A^{-1} \in (M_q^-)^{nxn}$, we get $\tau_{-j^+} AP_-(\tau_{j^+} A^{-1} f) \in \begin{pmatrix} 0 \\ L_1^- \\ \end{pmatrix}^n$, i.e. $T_j\varphi = P_+(f - \tau_{j^+} AP_-(\tau_{j^+} A^{-1} f)) = 0$, which proves the first part of our Lemma.

It remains to observe that $\dim N_j = \dim(\ker(H_j^+|_{\mathrm{Im} H_j^-})) = \dim \mathrm{Im} H_j^- -$ $-\dim \mathrm{Im} H_j^+|_{\mathrm{Im} H_j^-} = \dim \mathrm{Im} H_j^- - \dim \mathrm{Im} K_j$ and $\dim \mathrm{Im} H_j^- = \upsilon_- + nj^+$ (see [7]) to complete the proof.

We define the m.-f. $U = \tau_{\upsilon_-} P_-(\tau_{-\upsilon_-} B^{-1}) P_+(\tau_{-\upsilon_-} A^{-1})$ and square matrices $b_{m,k}^{(j)}$, $a_{m,k}^{(j)}$, $u_{m,k}^{(j)}$ $(m,k,j \in \mathbb{Z})$, given by $b_{m,k}^{(j)} = \langle B^{-1} \rangle_{-(m+k)-1-j^-}$, $a_{m,k}^{(j)} = \langle A^{-1} \rangle_{\upsilon_- + m - k}$ when $m < k$ and $a_{m,k}^{(j)} = 0$ when $m \geq k$, $u_{m,k}^{(j)} = \langle U \rangle_{-(m+k)-1-j^-}$, where for a m.-f. Φ by $\langle \Phi \rangle_k$ we mean the following matrix: $\langle \Phi \rangle_k = \frac{1}{2\pi i} \int_\Gamma t^{-k-1} \Phi(t) dt$. For $j > -\upsilon_-$ we define the block matrices A_j, B_j, U_j, $\mathfrak{K}_j$ by: $B_j = \| b_{m,k}^{(j)} \|_{m=0,\dots,j'-j^-}^{k=0,\dots,j^+ + \upsilon_- - 1}$, $A_j = \| a_{m,k}^{(j)} \|_{m=0,\dots,j^+ + \upsilon_- - 1}^{k=0,\dots,j^+ + \upsilon_- - 1}$, $U_j = \| u_{m,k}^{(j)} \|_{m=0,\dots,j'-j^-}^{k=0,\dots,j^+ + \upsilon_- - 1}$, $\mathfrak{K}_j = U_j + B_j A_j$, where $j' = \max\{j, \upsilon_+\}$. For $j \in \mathbb{Z}$ we also define mappings $\psi_j : C^{n(\upsilon_- + j^+)} \to \Im_{-(\upsilon_- + j^+)}$ by the formula $\psi_j q = \sum_{k=-(\upsilon_- + j^+)}^{-1} q_k t^k$, where $q = [q_{-(\upsilon_- + j^+)}, \dots, q_{-1}]$, $q_k \in \mathbb{C}^n$ $(k = -(\upsilon_- + j^+), \dots, -1)$.

The following statement is true:

Lemma 2. If $j \leq -\upsilon_-$, then $\dim N_j$=0. If $j > -\upsilon_-$, then $\dim N_j = \upsilon_- + nj^+ - r_j$, where $r_j = \mathrm{rang}\,\mathfrak{K}_j$. Besides, if $j \geq \upsilon_+$, then $\dim N_j = \upsilon_- + nj - n \deg q_{02}^- - \sum_{i=1}^n (\deg q_{i1}^- - \deg q_{i2}^-)$.

Proof. Since $p_{01}^+ \cdot p_{02}^+ \cdot \tau_{-\upsilon_-} \cdot A^{-1} \in (L_q^-)^{nxn}$ and $q_- B^{-1} \in (L_p^+)^{nxn}$, the following equalities are true:

$$\sum_{k=0}^{n_-} \langle A^{-1} \rangle_{m-k} \langle q_+ \rangle_k = 0, \; m = \upsilon_- + 1, \dots, \tag{2}$$

$$\sum_{k=0}^{\upsilon_+} <B^{-1}>_{m-k} <q_->_k = 0,\ m=-1,-2,\ldots \qquad (3)$$

Let $j\le -\upsilon_-$, $q\in\ker K_j$ and $\varphi = H_j^- q$. Then it is obvious, that $\varphi\in\Re^n\bigcap(L_p^+)^n$. According to Lemma 1, a v.-f. $\psi=\tau_j B^{-1}\varphi\in N_j$ and $\psi\in(L_p^+)^n$. Since $\varphi=\tau_{-j}B\psi$, then $<\varphi>_0=\ldots=<\varphi>_{-j-1}=0$. The last equalities mean that $\sum_{k=0}^{\upsilon_-}<A^{-1}>_{m+k}<q>_{-k}=0$ $(m=0,\ldots,\upsilon_- -1)$, since $\varphi=\sum_{m=-j-1}^{-\infty} z^m\sum_{k=1}^{\upsilon_-}<A^{-1}>_{m+k}<q>_{-k}$. Hence, using (2) and observing that $<q_+>_0\neq 0$ and $n_-\le\upsilon_-$, we obtain

$$<\varphi>_{-j}=\sum_{k=1}^{\upsilon_-}<A^{-1}>_{-j+k}<q>_{-k}=-\sum_{k=1}^{\upsilon_-}\sum_{i=1}^{n_-}\frac{<q_+>_i}{<q_+>_0}<A^{-1}>_{-j+k-i}<q>_{-k}=$$

$$=-\sum_{k=1}^{n-}\frac{<q_+>_i}{<q_+>_0}\sum_{k=1}^{\upsilon_-}<A^{-1}>_{-j+k-i}<q>_{-k}=0.$$

Similarly we get $<\varphi>_{-j+1}=<\varphi>_{-j+2}=\ldots=0$, i.e. $\varphi=0$. The Lemma 1 implies that $N_j=\{0\}$.

Let now $j>-\upsilon_-$ while $q\in\Im_{-(\upsilon_-+j^+)}$. Using $\tau\cdot P_-(\tau_{j^+-1}A^{-1})q\in\begin{pmatrix}0\\ L_q^-\end{pmatrix}^n$, $\tau_{j^++\upsilon_-}P_+(\tau_{-\upsilon_-}A^{-1})q\in(L_q^+)^n$ and equality $A^{-1}=\tau_{-j^++1}P_-(\tau_{j^+-1}A^{-1})+\tau_{\upsilon_-}P_-(\tau_{-j^++1-\upsilon_-}P_+(\tau_{j^+-1}A^{-1}))+$ $+\tau_{\upsilon_-}P_+(\tau_{-\upsilon_-}A^{-1})$, we obtain that $H_j^-q=\tau_{j^++\upsilon_-}P_+(\tau_{-\upsilon_-}A^{-1})q+g$, where

$$g(t)=P_+(\tau_{j^+}\cdot\tau_{\upsilon_-}P_-(\tau_{-j^++1-\upsilon_-}P_+(\tau_{j^+-1}A^{-1}))q)=\sum_{k=0}^{\upsilon_-+j^+-2}<g>_k t^k,$$

$<g>_k=\sum_{m=k-(\upsilon_-+j^+)+1}^{-1}<A^{-1}>_{k-m-j^+}<q>_m$. Hence, taking into account that $\tau_j\tau_{\upsilon_-}P_+(\tau_{-\upsilon_-}B^{-1})\ \tau_{\upsilon_-}P_+(\tau_{-\upsilon_-}A^{-1})q\in(L_1^+)^n$, we get that $H_j^+H_j^-q=P_-(\tau_{j^-}Uh)+H_j^+g$, where $h=\tau_{\upsilon_-+j^+}q$. Consequently, the condition $K_jq=0$ is equivalent to the following infinite system of equalities:

$$\sum_{k=0}^{\upsilon_-+j^+-1}<U>_{m-k-j^-}<h>_k+\sum_{k=0}^{\upsilon_-+j^+-2}<B^{-1}>_{m-k-j^-}<g>_k=0,\ m=-1,-2,\ldots$$

It is easy to see that $(<h>_0^T,\ldots,<h>^T_{\upsilon_-+j^+-1})^T=\psi_j^{-1}q$ and $(<g>_0^T,\ldots,<g>^T_{\upsilon_-+j^+-2},0)^T=A_j\psi_j^{-1}q$. The remark above implies that the condition $q\in\ker K_j$ is equivalent to the equality $U_j\psi_j^{-1}q+B_jA_j\psi_j^{-1}q=0$. By writing the last equality in the following form $\mathfrak{K}_j\psi_j^{-1}q=0$, we finally obtain that $K_jq=0$, iff $\mathfrak{K}_j\psi_j^{-1}q=0$. Consequently, $\dim\ker K_j=\dim\ker\mathfrak{K}_j$. In view of (1) the following equality is true:

$\dim N_j = \upsilon_- + nj^+ - \dim \operatorname{Im} K_j = \upsilon_- + nj^+ - (n(\upsilon_- + j^+) - \dim\ker K_j) = \upsilon_- + nj^+ - r_j$.

It remains to prove the last statement of our Lemma 2. Let $j \ge \upsilon_+$ and $q \in \Im_j$, then $\varphi = \tau_{-j} A q \in (\overset{0}{L_p^-})^n$, $\varphi \in \overset{0}{D_p^-}(A^{-1})$ and $H_j^- \varphi = q$. Consequently, $\Im_j \subset \operatorname{Im} H_j^-$. A v.-f. $y \in D_q^+(B^{-1})$ we write as follows $y = \tilde{q} + q_- y_0$, where $y_0 \in D_q^+(B^{-1})$, while $\tilde{q}$ is a vector polynomial, whose degree does not exceed $\upsilon_+ - 1$. We have $q_- B^{-1} y_0 \in (L_1^+)^n \bigcap L_q^n$ (i.e. $q_- B^{-1} y_0 \in (L_q^+)^n$), and, therefore, the equality $B^{-1} y = B^{-1}\tilde{q} + q_- B^{-1} y_0$ implies that $H_0^+ y = H_0^+ \tilde{q}$, i.e. $\operatorname{Im} H_0^+ = \operatorname{Im}\left(H_0^+ |_{\Im_{\upsilon_+}} \right)$.

The operator $T' : \overset{0}{D_q^-}(B) \to (\overset{0}{L_q^-})^n$ is defined by formula $T'y = P_-(By)$. We prove that $\ker T' = \operatorname{Im} H_0^+$. Assume that $\varphi \in \operatorname{Im} H_0^+$. Then there exists $y \in \Im_{\upsilon_+}$ such that $\varphi = H_0^+ y = P_-(B^{-1} y)$. Now $B\varphi = y - BP_+(B^{-1} y)$, $BP_+(B^{-1}y) \in (L_1^+)^n$ implies $B\varphi \in (L_1^+)^n \bigcap L_q^n$, i.e. $\varphi \in \ker T'$. Conversely, if $\varphi \in \ker T', \varphi \in (\overset{0}{L_q^-})^n$ and $B\varphi = \psi \in (L_q^+)^n$, i.e. $\varphi = B^{-1}\psi = P_-(B^{-1}\psi) = H_0^+ \psi$. B admits a left factorization in L_q (see [4]), and, as it is known, $\dim\ker T'$ coincides with the sum of positive partial indices of B. Since B is analytic inside the circle, then its partial indices are nonnegative (see [4]). Therefore, $\dim\ker T'$ coincides with total index of B. On the other hand, total index of B is equal to the number of zeros inside Ω_+ (by taking into account their multiplicities) of function $\det B$. Thus, $\dim \operatorname{Im} H_0^+ = n \deg q_{02}^- + \sum_{i=1}^{n} (\deg q_{i1}^- - \deg q_{i2}^-)$. For $j \ge 0$ we obtain

$$\operatorname{Im} H_0^+ \supset \operatorname{Im} K_j = \operatorname{Im}\left(H_0^+ |_{H_j^-(\Im_{-(\upsilon_- + j)})} \right) = \operatorname{Im}\left(H_0^+ |_{\operatorname{Im} H_0^-} \right) \supset \operatorname{Im}\left(H_0^+ |_{\Im_j} \right),$$

and for $j \ge \upsilon_+$ we get $\operatorname{Im} H_0^+ \supset \operatorname{Im} K_j \supseteq \operatorname{Im} H_0^+$. Consequently, we have $\dim \operatorname{Im} K_j = \dim \operatorname{Im} H_0^+$, $j \ge \upsilon_+$, and the Lemma is proved.

Note that, particularly, the following statement is proved:

Corollary 1. For $j > -\upsilon_-$ the following equality is true: $\ker K_j = \psi_j \ker \mathfrak{K}_j$.

Theorem 1. The partial indices of m.-f. G can be calculated by formulas:

$$\kappa_i = -\upsilon_- + \chi_0 + \operatorname{card}\{ j : n\theta_j - r_j + r_{j-1} < i, \quad j = -\upsilon_- + 1, \ldots, \upsilon_+ + 2) \}, \qquad (4)$$

where $r_{-\upsilon_-} = \upsilon_-$, $r_j = \operatorname{rang} \mathfrak{K}_j$ $(j > -\upsilon_-)$ and $\theta_j = 1,\ j > 0, \theta_j = 0,\ j \le 0$.

Proof. It is known that $\dim N_j$ is equal to the sum of negative partial indices of the m.-f. $\tau_{-(\chi_0 + j)} G$ with the minus sign. The partial indices of the m.-f. $\tau_{-(\chi_0 + j)} G$ are equal to $\kappa_i - \chi_0 - j$ $(i = 1, \ldots, n)$. As we have $j > \kappa_1 - \chi_0$, then $\dim N_j > 0$. Consequently, $-\upsilon_- \le \kappa_1 - \chi_0$. Similarly (see [5]), it is not difficult to see that

$$\kappa_i - \chi_0 = \eta_- + \text{card}\{j \,|\, \dim N_j - \dim N_{j-1} < i,\; j = \eta_- + 1, \ldots, \eta_+\},$$

where η_-, η_+ are arbitrary integer numbers satisfying to $\eta_- \le \kappa_1 - \chi_0 \le \ldots \ldots \le \kappa_n - \chi_0 \le \eta_+$. We can take η_- to be equal to $-\upsilon_-$, while by Lemma 2 we can choose η_+ to be the number $\upsilon_+ + 2$. Taking into account also the equality $\dim N_j - \dim N_{j-1} = n\theta_j + r_{j-1} - r_j$, we get (4).

Lemmas 1 and 2 imply that $N_j = \{\tau_{j^-} B^{-1} P_+(\tau_{j^+} A^{-1} \psi_j q),\; q \in \ker \mathfrak{K}_j\},\; j > -\upsilon_-$. We denote $\widehat{N}_j = N_j + \tau N_j = \{\varphi + \tau\psi; \varphi, \psi \in N_j\}$ and $N_j(0) = \{\varphi(0), \varphi \in N_j\}$. It is known (see [3]) that $\widehat{N}_j \subset N_{j+1}, j \in \mathbb{Z}$. We denote by M_j some direct complement of $\widehat{N}_j$ in N_{j+1}. Spaces M_j (see [2]) are called (p,j)-index subspaces. We denote $\xi_i = \kappa_i - \chi_0 + 1\ (i = 1, \ldots, n)$. It is known that $N_j = \{0\}$ for $j \le \xi_1 - 1$ and $N_j = \widehat{N}_{j-1}$ for all $j \in \mathbb{Z} \setminus \{\xi_1, \ldots, \xi_n\}$. The following statement follows from [3]:

Proposition 1. Assume $\varphi_{i1}, \ldots, \varphi_{im_i}\ (i = 1, \ldots, n,\; m_i \in \mathbb{N})$ are bases in the space $M_{\xi_i - 1}$. Then $\tau_{-\chi_0} G = G_- \Lambda G_+^{-1}$, where $G_+ = [\varphi_{11}, \ldots, \varphi_{1m_1}, \varphi_2, \ldots, \varphi_{2m_2}, \ldots, \varphi_{n1}, \ldots, \varphi_{nm_n}]$, $\Lambda = \text{diag}[\tau_{\xi_1 - 1}, \ldots, \tau_{\xi_n - 1}]$ and $G_- = G_0 G_+ \Lambda^{-1}$ is a factorization of a m.-f. $\tau_{-\chi_0} G$.

Assume that $K'_j = [< A^{-1} >_{\upsilon_- - j^-}, \ldots, < A^{-1} >_{1-j}]$. We call the vectors $q_i = [q_i^1, \ldots, q_i^{\upsilon_- + \xi_i^+}]$ $(q_i^k \in \mathbb{C}^n, i = 1, \ldots, n;\; k = 1, \ldots, \upsilon + \xi_i^+)$ a factorization collection for the m.-f. *G*, if $\mathfrak{K}_{\xi_i} q_i = 0,\; i = 1, \ldots, n$, while vectors $K'_{\xi_1} q_1, \ldots, K'_{\xi_n} q_n$ are linearly independent.

Proposition 2. A m.-f. *G* possesses a factorization collection.

Proof. Let $\varphi \in N_j$, then there exists a vector $q = [q_{-(\upsilon_- + j^+)}, \ldots, q_{-1}] \in \ker \mathfrak{K}_j$, $q_s \in \mathbb{C}^n, s = (-\upsilon_- + j^+, \ldots, -1)$, such that $\varphi(t) = t^{-j^-} B^{-1}(t) P_+(\tau_{j^+} A^{-1} \psi_j q)$.

$$< \tau_{j^+} A^{-1} \psi_j q >_m = \frac{1}{2\pi i} \int_\Gamma t^{j^+} A^{-1}(t) \sum_{k=-\upsilon_- + j^+}^{-1} q_k t^k t^{-m-1} dt = \sum_{k=-\upsilon_- + j^+}^{-1} \frac{q_k}{2\pi i} \int_\Gamma A^{-1}(t) t^{j^+ + k - m - 1} dt =$$

$$= \sum_{k=-\upsilon_- + j^+}^{-1} < A^{-1} >_{m-k-j^+} q_k.$$

The v.-f. φ is analytic in Ω_+, and hence, it can be de expanded into the series $\varphi(t) = B^{-1}(t) \sum_{m=0}^{\infty} \left(\sum_{k=-\upsilon_- + j^+}^{-1} < A^{-1} >_{m-k-j^+} q_k \right) t^{m+j^-}$ in a neighborhood of 0. Besides, $N_j(0) = \{B^{-1}(0) K'_j q, q \in \ker \mathfrak{K}_j\}$, since the m.-f. $B(0)$ is invertible. The existence of a factorization collection follows now from properties of spaces $N_j(0)$ (see [3]). Proof is completed.

Theorem 2. Let $q_i (i = 1, \ldots, n)$ be a factorization collection for the m.-f. *G* and $\varphi_i = \tau_{\xi_i^-} B^{-1} H_{\xi_i}^- \psi_{\xi_i} q_i,\; i = 1, \ldots n$, then $G_+ = [\varphi_1, \ldots, \varphi_n]$, $\Lambda = \text{diag}[t^{\kappa_1}, \ldots, \tau^{\tau_n}]$,

$G_- = GG_+\Lambda^{-1}$ is a factorization of m.-f. G.

Proof. Lemma 1 and Corollary 1 imply that $\varphi_i \in N_{\xi_i}(i = 1,\ldots,n)$. Since $\varphi_1(0),\ldots,\varphi_n(0)$ are linearly independent, then φ_i does not belong to $\widehat{N}_{\xi_i-1}$. Consequently, $\varphi_i \in M_{\xi_i-1}(i = 1,\ldots,n)$. Taking into account linear independence of a v.-f. $\varphi_i\ (i = 1,\ldots,n)$ we deduce the proof of our Theorem from the Proposition 1.

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