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ON AUTOMORPHISMS OF PERIODIC PRODUCTS OF GROUPS

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In this paper it has been proved that each normal automorphism of the n-periodic product of cyclic groups of odd order $r \geq 1003$ is inner, whenever r divides n.

Keywords: n-periodic product of groups, normal, automorphism, inner automorphism, free Burnside group.

Introduction. The present paper studies the normal automorphisms of n-periodic products of finite cyclic groups. The operations of n-periodic products were constructed in the paper [1] for any odd number $n \geq 665$. They posses many properties of the classical operations of free and direct products of groups, including exactness, associativity and hereditary property for subgroups (see [2]). The papers [3–5] are devoted to the study of some other properties of n-periodic products of groups. First we give some definitions. Suppose G is an arbitrary group and $\mathscr{N} = \mathscr{N}(G)$ is the set of all normal subgroups of G. Consider the set

$$Aut_{\mathscr{N}}(G) \risingdotseq \{\varphi \in Aut(G) |\, \varphi(H) = H \text{ for all } H \in \mathscr{N}\}.$$

Each automorphism from $Aut_{\mathscr{N}}(G)$ is called *normal* automorphism. It is easy to see that $Inn(G) \trianglelefteq Aut_{\mathscr{N}}(G) \leq Aut(G)$, where $Inn(G)$ is the group of all inner automorphisms of G. It is clear that if φ is a normal automorphism of group G, then it induces some automorphism of quotient group G/N.

A. Lyubotzky [6] proved that every normal automorphism of a free product of infinite cyclic groups is inner, i.e. the equality $Inn(F) = Aut_{\mathscr{N}}(F)$ is true, where

$$F = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}.$$

The equality $Inn(G) = Aut_{\mathscr{N}}(G)$ was proved by different authors for various groups G (see [7–13]. For example, Minasyan and Osin in [12] proved, that if G is a non-cyclic relatively hyperbolic group without non-trivial finite normal subgroups, then $Inn(G) = Aut_{\mathscr{N}}(G)$. In the paper [13] it is proved that all normal automorphisms of free Burnside group $B(m,n)$ of rank $m > 1$ and odd period $n \geq 1003$ are

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inner. Improving the result of [6], M.V. Neshchadim in [11] proved that any normal automorphism of a free product of nontrivial groups is inner. Note that the analogous statement is false in general for n-periodic products of groups. This was shown in [4].

The main result of this paper is the following theorem.

Theorem 1. Let $F = \prod_{i\in I}^{n}\langle a_i\rangle$ be an n-periodic product of cyclic groups $\langle a_i\rangle$ of odd order $r \geq 1003$, where r divides n. Then each normal automorphism ϕ of the group F is inner automorphism.

Some Quotient Groups of F and Auxiliary Lemmas. In the proof of Theorem 1 we need some quotient groups of the group F. For simplicity we denote generators a_1 and a_2 by a and b respectively. Suppose $F(n,0)$ is a free group with generators $a_1 = a$, $a_2 = b$, a_3 .... For any natural $\beta > 0$ by $F(n,\beta)$ denote a group with the same generators and system of defining relations $\{a_i^r = 1,\, i \in I\}$, and $\{A^n = 1$, where $A \not\equiv a_i$ for each $i \in I$ and $A \in \bigcup_{i\leq\beta} \mathscr{E}_i\}$ (see [16, ch. VI, §2]. For $\alpha = 0, 1, 2$ we denote $\Gamma_\alpha \rightleftharpoons F(n,\alpha)$. Suppose $\alpha > 2$ and the groups Γ_δ are already defined for $\delta < \alpha$. Let Ψ_α be the set of all elementary periods C of rank $\alpha - 1$, satisfying the relation

$$C \overset{\alpha-2}{=} A^{-d}Z^{-1}B^{-d}ZA^{d}Z^{-1}B^{d}Z, \tag{1}$$

where A and B are minimized elementary periods of some ranks γ and β respectively, $Z \in \mathscr{M}_{\alpha-2}, \gamma \leqslant \beta \leqslant \alpha-2, d = 191$ (see [14, §1]). We choose the subset $\overline{\Psi}_\alpha \subset \Psi_\alpha$, so that each element $C \in \Psi_\alpha$ is conjugate to exactly one word D in the group $\Gamma_{\alpha-2}$, satisfying $D \in \overline{\Psi}_\alpha$ or $D^{-1} \in \overline{\Psi}_\alpha$. We denote by Φ_α the set of words for each period $C \in \overline{\Psi}_\alpha$ and fixed elementary period A of rank $\gamma \leq \alpha - 2$ from (1) containing exactly two words

$$C^{200}AC^{200}A^{2}\cdots A^{r-1}C^{200}x_c, \tag{2}$$

$$C^{300}AC^{300}A^{2}\cdots A^{r-1}C^{300}y_c, \tag{3}$$

where the elements x_c and y_c are chosen so that one of them is equal to a and the other one is equal to b. Obviously, for each C there are two possibilities (a,b) and (b,a) for the choice of the pair (x_c,y_c). If a concrete pair is not mentioned, then we will assume it is chosen arbitrarily. In the rest we will point out concrete values of some pairs (x_c,y_c) (see the definition of $\mathcal{K}'$).

We consider groups

$$\Gamma_\alpha \rightleftharpoons \left\langle a_1,a_2,\ldots \mid a_i^r = 1, R^n = 1, F = 1, R \in \bigcup_{\beta\leq\alpha} \mathscr{E}_\beta, R \not\equiv a_i, i \in I, F \in \bigcup_{\beta\leq\alpha} \Phi_\beta \right\rangle$$

and

$$\Gamma \rightleftharpoons \left\langle a_1,a_2,\ldots \mid a_i^r = 1, R^n = 1, F = 1, R \in \bigcup_{\beta>0} \mathscr{E}_\beta, R \not\equiv a_i, i \in I, F \in \bigcup_{\beta\geq 3} \Phi_\beta \right\rangle.$$

By $\mathcal{K}$ we denote the class of all groups Γ, which are obtained by the above described method for different choices of the subset $\overline{\Psi}_\alpha \subset \Psi_\alpha$ and elements x_c, y_c. The following lemmas can be proved exactly in the same way as the analogous statements of papers [13–15].

Lemma 1. Each $\Gamma \in \mathcal{K}$ is an infinite group, whose each two non-commuting elements generate the whole group Γ.

Lemma 2 (see Lemma 13, [13]). If $X^{\delta} \overset{F}{=} T X^{\varepsilon} T^{-1}$, then the subgroup $\langle X, T\rangle_F$ is cyclic.

Lemma 3 (see Lemma 3, [15]). If E is an elementary period of rank γ, $Z_1, Z_2 \in \mathscr{M}_{\lambda} \cap \mathscr{A}_{\lambda+1}$ for some $\lambda \geq \gamma$, $\left[E^d,\, Z^{-1} E^d Z\right] \neq 1$ and the commutators $\left[E^d,\, Z^{-1} E^d Z\right]$ and $\left[E^d,\, Z'^{-1} E^d Z'\right]$ are conjugate in F, then for some integers u and v either $Z'^{-1} E^{-d} Z' = E^u Z^{-1} E^{-d} Z E^v$ or $Z' E^{-d} Z'^{-1} = E^u Z^{-1} E^d Z E^v$ in F.

Lemma 4 (see Lemma 4, [15]). For $1 \leq |k| \leq (r-1)/2$ each of the commutators

$$\left[a^k,\, b^{-9} a^k b^9\right] \equiv a^{-k} b^{-9} a^{-k} b^9 a^k b^{-9} a^k b^9$$

is minimized elementary period of rank 2.

To obtain the quotient groups that we will use later in the proof of Theorem 1 , we will add some extra conditions on the set $\overline{\Psi}_3$ of groups from the class $\mathcal{K}$.

Obviously, a is a minimized elementary period of rank 1 and $b^9 \in \mathscr{M}_1$. According to Lemma 4 and definition of the set Ψ_3, we have $\left[a^d,\, b^{-9} a^d b^9\right] \in \Psi_3$. Denote by $\mathcal{K}'$ the set of all groups $\Gamma \in \mathcal{K}$, for which the following conditions hold:
1. the set $\overline{\Psi}_3$ in the definition of the group Γ is chosen so that $\left[a^d,\, b^{-9} a^d b^9\right] \in \overline{\Psi}_3$;
2. for period $C \rightleftharpoons \left[a^d,\, b^{-9} a^d b^9\right] \in \overline{\Psi}_3$ the elements x_c and y_c, appearing in (2), (3) respectively, are chosen to be $x_c = b$, $y_c = a$;
3. for all the other periods $C \in \overline{\Psi}_{\alpha}$, $C \overset{\alpha-2}{\neq} \left[a^d,\, b^{-9} a^d b^9\right]$ the elements x_c and y_c, appearing in (2) and (3) respectively, are chosen as $x_c = a$, $y_c = b$.

The following lemmas can be proved exactly the same way as the analogous statements of paper [13].

Lemma 5 (see Lemma 7, [13]). For every $\Gamma \in \mathcal{K}'$ the relations

$$\left[a^d,\, b^{-9} a^d b^9\right]^{200} a \left[a^d,\, b^{-9} a^d b^9\right]^{200} a^2 \cdots a^{(r-1)} \left[a,\, b^{-9} a b^9\right]^{200} b = 1, \tag{4}$$

$$\left[a^d,\, b^{-9} a^d b^9\right]^{300} a \left[a^d,\, b^{-9} a^d b^9\right]^{300} a^2 \cdots a^{(r-1)} \left[a,\, b^{-9} a b^9\right]^{300} a = 1 \tag{5}$$

and

$$C^{200} A C^{200} A^2 \cdots A^{n-1} C^{200} a = 1,\ C^{300} A C^{300} A^2 \cdots A^{r-1} C^{300} b = 1$$

for each period $C \in \overline{\Psi}_{\alpha}$ and $C \overset{\alpha-2}{\neq} \left[a^d,\, b^{-9} a^d b^9\right]$.

Lemma 6 (see Lemma 11, [13]). Let $a, b \in \{a_i\}$, $i \in I$, $\phi : \prod\limits_{i \in I}^{n} \langle a_i \rangle \to \prod\limits_{i \in I}^{n} \langle a_i \rangle$ be a normal automorphism and let $\phi(Z) = b^9$. Then the commutator $[a^d, Z^{-1} a^d Z]$ is not a conjugate of $[a^d,\, b^{-9} a^d b^9]^{-1}$ in the group $\prod\limits_{i \in I}^{n} \langle a_i \rangle$.

Lemma 7 (see Lemma 8, [13]). If in the group $\Gamma \in \mathcal{K}'$ the relation

$$\left[a^k,\, b^{-9}a^kb^9\right]^s a^t \left[a^k,\, b^{-9}a^kb^9\right]^s a^{2t}\cdots a^{(r-1)t}\left[a^k,\, b^{-9}a^kb^9\right]^s a^t = 1$$

holds, where $1 \le |k|,\ |t| \le (r-1)/2$, $k \equiv d\cdot t(\bmod r)$ and $q+2 \le s \le (r-1)/2-2$, then $k = \pm d$ and $t = \pm 1$.

Properties of Normal Automorphisms of F.

Lemma 8. Let $F = \prod\limits_{i\in I}^{n}\langle a_i\rangle$ be a n-periodic product of cyclic groups $\langle a_i\rangle$ of odd order $r \ge 1003$, where r divides n. If ϕ is a normal automorphism of F, then $\phi(a_i) = u_i a_i^{s_i} u_i^{-1}$ for some $u_i \in F$, and some s_i satisfying $(s_i, r) = 1$, $i \in I$.

Proof. The operations of n-periodic product for odd $n \ge 665$ are exact. Therefore, the elements a_i, $i \in I$, have order r in F. Hence, their automorphic images $\phi(a_i)$, $i \in I$, are also of order r.

Since ϕ is normal automorphism, then we have the equalities

$$N_{a_i} = \phi(N_{a_i}) = N_{\phi(a_i)} = N_{ua_ju^{-1}} = N_{a_j},$$

where N_x stands for the normal closure of element x. In the light of the obvious relation $a_i \in N_{a_i}$ we get $a_i \in N_{a_j}$. The sum of degrees of the letter a_i in any word from the normal closure N_{a_j} is equal to 0 modulo r. Indeed, any defining relation of group the F has the form either a_i^r or A^n, where $A \in F$ is an elementary period of some rank. Thus, the sum of degrees of occurrences of the letter a_i in each word from the normal closure N_{a_j} has the form $ur + vn$, which is a multiple of r by the hypothesis of Lemma.

Assuming that $j \ne i$, we obtain that a_i is equal to some element from the normal closure N_{a_j}, the sum of degrees of the letter a_i in which is equal to 0 modulo r. Thus, we get an obvious contradiction. So, we can conclude that $j = i$. Consequently, we have proved that $\phi(a_i) = ua_i^s u^{-1}$ for some integer s. Applying the automorphism ϕ^{-1} to both sides of this equality, we get ${a_i}^{ss_1} = a_i$ for some integer s_1. This implies that $ss_1 \equiv 1(mod\ r)$.

The Lemma is proved.

Lemma 9. Let $a, b \in \{a_i\}$, $i \in I$, $\phi : F \to F$ be a normal automorphism and let $\phi(a) = a^t$, $\phi(b) = ub^tu^{-1}$. Fix an element Z such that $\phi(Z) = b^9$. Then the commutators $[a^d, Z^{-1}a^dZ]$ and $[a^d,\, b^{-9}a^d b^9]$ are conjugate in the group $\prod\limits_{i\in I}^{n}\langle a_i\rangle$.

Proof. We will prove the Lemma by contradiction. Assume that the commutators $[a^d, Z^{-1}a^dZ]$ and $[a^d,\, b^{-9}a^d b^9]$ are not conjugate in the group $F = \prod\limits_{i\in I}^{n}\langle a_i\rangle$. Since $\phi(Z) = b^9$, we obtain $\phi(\left[a^d, Z^{-1}a^dZ\right]) = \left[a^d,\, b^{-9}a^d b^9\right]$ in $\prod\limits_{i\in I}^{n}\langle a_i\rangle$. Then, according to [16, ch. VI, §2, i. 4] and [16, ch. IV, §3, i. 12], one can assume that $Z \in \mathscr{M}_\alpha \cap \mathscr{A}_{\alpha+1}$ for some $\alpha \ge 1$. Choose a reduced form G_1 of the commutator $[a^d, Z^{-1}a^dZ]$ according to Lemma 3.2 of [14]. By the definition of reduced form we have $G_1 \overset{0}{=} w[a^d, Z^{-1}a^dZ]w^{-1}$ for some $w \equiv a^j$. By Lemma 7.2 of [14] G_1 is an elementary period of some rank $\delta \ge 2$ for each $\Gamma \in \mathcal{K}$. Since, by assumption the commutators $[a^d, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^9]$ are not conjugate in the group $\prod\limits_{i\in I}{}^{n}\langle a_i\rangle$, using Lemma 11 [13], we obtain that the elements $[a^d, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^9]^{\pm 1}$ are not conjugate in the group Γ_1. Therefore, there exist groups from class $\mathcal{K}' \subset \mathcal{K}$ such that $G_1 \in \overline{\Psi}_{\delta+1}$.

Let Γ^+ be one of such groups. By Lemma 5, the relations (4), (5) and

$$G_1^{200}aG_1^{200}a^2\cdots a^{(r-1)}G_1^{200}a = 1, \tag{6}$$

$$G_1^{300}aG_1^{300}a^2\cdots a^{(r-1)}G_1^{300}b = 1 \tag{7}$$

hold in the group Γ^+.

Since $G_1 \overset{0}{=} a^j[a^d, Z^{-1}a^dZ]a^{-j}$, we get $\phi(G_1) \overset{\prod_{i\in I}^{n}\langle a_i\rangle}{=} a^j[a^d,\ b^{-9}a^db^9]a^{-j}$. From the definition of the group Γ^+, for some normal subgroup N of the group $\prod\limits_{i\in I}{}^{n}\langle a_i\rangle$ we have $\Gamma^+ = \prod\limits_{i\in I}{}^{n}\langle a_i\rangle/N$. Applying ϕ to both sides of the relation (6), we obtain

$$(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}a^t(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}\cdots$$
$$\cdots a^{(r-1)t}(a^{jt}[a^k,b^{-9}a^kb^9]a^{-jt})^{200}a^t \in N.$$

Therefore,

$$[a^k,b^{-9}a^kb^9]^{200}a^t[a^k,b^{-9}a^kb^9]^{200}\cdots a^{(r-1)t}[a^k,b^{-9}a^kb^9]^{200}a^t \in N,$$

that is

$$[a^k,b^{-9}a^kb^9]^{200}a^t[a^k,b^{-9}a^kb^9]^{200}a^{2t}\cdots a^{(r-1)t}[a^k,b^{-9}a^kb^9]^{200}a^t \overset{\Gamma^+}{=} 1.$$

From here, by Lemma 7 we obtain that $k = \pm d$ and $t = \pm 1$.

In the case $t = 1$ we have $\phi(a) = a$ and $\phi(G_1) = a^j[a^d,\ b^{-9}a^db^9]a^{-j}$ in $\prod\limits_{i\in I}{}^{n}\langle a_i\rangle$. Applying ϕ to both sides of the relation (7), we obtain

$$(a^j[a^d,b^{-9}a^db^9]a^{-j})^{300}a\cdots a^{(r-1)}(a^j[a^d,b^{-9}a^db^9]a^{-j})^{300}u^{-1}bu \in N.$$

Therefore,

$$[a^d,b^{-9}a^db^9]^{300}a[a^d,b^{-9}a^db^9]^{300}a^2\cdots a^{(r-1)}[a^d,b^{-9}a^db^9]^{300}a^{-j}u^{-1}bua^j \overset{\Gamma^+}{=} 1.$$

Using the last equality and (5), we immediately deduce that the equality $a = a^{-j}u^{-1}bua^j$ holds in the group Γ^+, that is $a \overset{\Gamma^+}{=} u^{-1}bu$. Thus, $\phi(a) \overset{\Gamma^+}{=} \phi(b)$ and hence $\phi(a^{-1}b) \in N$. Since $\phi(N) = N$, we obtain that $a^{-1}b \in N$, which implies that Γ^+ is a finite cyclic group. This contradicts to infiniteness of Γ (see Lemma 2). The case $t = -1$ can be disproved in a similar way, using the relations of the form (5).

Proposition 1. Suppose $a,b \in \{a_i\}$, $i \in I$, $\phi : \prod\limits_{i\in I}{}^{n}\langle a_i\rangle \to \prod\limits_{i\in I}{}^{n}\langle a_i\rangle$ is a normal automorphism satisfying $\phi(a) = a^t$, $\phi(b) = ub^tu^{-1}$. Let us fix an element Z such that $\phi(Z) = b^9$. If the commutators $[a^d, Z^{-1}a^dZ]$ and $[a^d,\ b^{-9}\,a^d\,b^9]$ are conjugate in the group $\prod\limits_{i\in I}^{n}\langle a_i\rangle$, then for some integers p,s,l,r we have $Z=a^p b^9 a^s$, $t=1$ and $u=b^l a^r$.

Proof. Since the commutators $[a^d, Z^{-1}a^d Z]$ and $[a^d,\, b^{-9}a^d, b^9]$ are conjugate, in virtue of Lemma 3 , we obtain that for some integers r and s either

$$Z^{-1}a^{-d}Z \overset{\prod_{i\in I}^{n}\langle a_i\rangle}{=} a^r b^{-9}a^{-d}b^9 a^s$$

or

$$Za^{-d}Z^{-1} \overset{\prod_{i\in I}^{n}\langle a_i\rangle}{=} a^r b^{-9}a^{d}b^9 a^s.$$

Consider each of these cases:

A. Let $Za^{-d}Z^{-1}=a^r b^{-9}a^d b^9 a^s$ in $\prod\limits_{i\in I}^{n}\langle a_i\rangle$. Then $a^s Za^{-d}Z^{-1}a^{-s}=a^{s+r}b^{-9}a^d b^9$. If $s+r\not\equiv 0(\bmod r)$, then the word $a^{s+r}b^{-9}a^{-d}b^9$ is an elementary period of rank 2. Thus, the elementary period $a^{s+r}b^9 a^{-d}b^{-9}$ of rank 2 is conjugate to some power of a, which contradicts to Lemma 6.6 from [14]. If $s+r\equiv 0(\bmod r)$, we obtain that a^{-d} and a^d are conjugate, which contradicts Lemma 2. Therefore, the case A is impossible.

B. Let $Z^{-1}a^{-d}Z=a^r b^{-9}a^{-d}b^9 a^s$ in $\prod\limits_{i\in I}^{n}\langle a_i\rangle$. Repeating the reasoning of the previous case, we get $s+r\equiv 0(\bmod r)$ and $a^s Z^{-1}a^{-d}Za^{-s}=b^{-9}a^{-d}b^9$ in $\prod\limits_{i\in I}^{n}\langle a_i\rangle$. This means that the element $b^9 a^s Z^{-1}$ belongs to the centralizer of the element a^{-d} in the group $\prod\limits_{i\in I}^{n}\langle a_i\rangle$. Applying Theorem 5 of [1], we get $Z=a^p b^9 a^s$ for some integer p. Next we prove that $u=b^l a^r$. Applying ϕ to both sides of the equality $Z=a^p b^9 a^s$, we obtain $b^9=a^{pt}u^{-1}b^{9t}ua^{st}$. Now applying the homomorphism $\alpha: F\to F$ defined by formulae $\alpha(a)=a$, $\alpha(b)=1$ to both sides of the last equality, we get $a^{pt+st}=1$. Since $(t,r)=1$, then $p\equiv -s(\bmod r)$. Thus, $a^p u^{-1}$ belongs to normalizer of the element b^9, which, according to Lemma 1, implies $u=b^l a^p$ for some integer l. It remains to show that $t=1$. Note that from the equalities $b^9=a^{pt}u^{-1}b^{9t}ua^{-pt}$ and $u=b^l a^{pt}$ we have $b^9=b^{9t}$. Hence,

$$\phi(\left[a^d, b^{-9}a^d b^9\right]) \overset{B(m,n)}{=} a^{-pt}\left[a^k,\, b^{-9}a^k b^9\right]a^{pt}$$

for some $k\equiv d\cdot t\,(\bmod\, r)$, $(k,\, r)=1$ and $1\le |k|\le (r-1)/2$.

Suppose that Γ is one of the groups from class $\mathcal{K}'$ and $\Gamma=F/N$. Applying the normal automorphism ϕ to the left part of the relation (5) and conjugating the obtained element by a^{pt}, we get

$$\left[a^k,\, b^{-9}a^k b^9\right]^{300} a^t \left[a^k,\, b^{-9}a^k b^9\right]^{300} a^{2t}\cdots a^{(r-1)t}\left[a^k, b^{-9}a^{-k}b^9\right]^{300},\ a^t\in N.$$

From here, by Lemma 7 , it follows that $k=\pm d$ and $t=\pm 1$. Comparing the equality $b^9=b^{9t}$ with $t=\pm 1$, we deduce that $t=1$. The Lemma is proved.

The Proof of Theorem 1. Let $\phi : \prod\limits_{i\in I}{}^{n}\langle a_i\rangle \to \prod\limits_{i\in I}{}^{n}\langle a_i\rangle$ be a normal automorphism and $a,b \in \{a_i\}$, $i \in I$, are such elements that $\phi(a) = a^t$, $\phi(b) = ub^tu^{-1}$. Fix an element Z with $\phi(Z) = b^9$. According to Lemma 6, the commutators $[a^d, Z^{-1}a^dZ]$ and $[a^d, b^{-9}a^d b^9]$ are not conjugate in the group $\prod\limits_{i\in I}{}^{n}\langle a_i\rangle$.

Thus, according to Proposition 1 we get $Z = a^pb^9a^s$, $t = 1$ and $u = b^la^k$ for some integers p,s,l,k. This means that $\phi(a) = a$ and $\phi(b) = a^kba^{-k}$. Suppose a_j is one of the generators of the group F, different from a and b. Arguing as above, we can state that $\phi(a_j) = a^sa_ja^{-s}$ for some $s \in \mathbb{Z}$. It remains to prove that $k \equiv s(\bmod r)$. Let us multiply the automorphism ϕ with inner automorphism generated by the element a^{-k}. We obtain a new normal automorphism ϕ_1, satisfying conditions $\phi_1(a) = a$, $\phi(b) = b$ and $\phi_1(a_j) = a^{s-k}a_ja^{k-s}$. Applying the Proposition 1 to the pair b,a_j, we obtain that for some integer m the relation $a^{s-k}a_ja^{-(s-k)} = b^ma_jb^{-m}$ holds in the group F.Finally using this and Lemma 2, we obtain the equalities $a^{s-k} = b^m = a_j^l$ in F for some integer l. But the latter is possible only if $s-k \equiv m \equiv l \equiv 0(\bmod r)$.

This completes the proof of Theorem 1.

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