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INITIAL-BOUNDARY VALUE PROBLEM FOR SECOND ORDER DEGENERATE PSEUDOHYPERBOLIC EQUATIONS

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The paper studies a initial-boundary value problem for a class of second order degenerate pseudohyperbolic equations.

We prove the existence and uniqueness of the problem in the appropriately constructed functional space.

Keywords: pseudohyperbolic degenerate equation, weak solution.

Introduction. We consider the following initial-boundary value problem of Sobolev type

$$\begin{cases} L\left(\dfrac{\partial^2 u}{\partial t^2}\right) + M(u) = 0, \\ u\big|_{t=0} = u^{(0)}(x),\ u_t\big|_{t=0} = u^{(1)}(x), \\ u\big|_{\Gamma^*} = 0, \quad t > 0, \end{cases} \tag{*}$$

where $\Gamma^* \subset \Gamma = \partial\Omega$, L and M are differential operators to be perused later.

We are interested the case, when the elliptic operator L can be degeneration the part of the initial hyperplane.

We treat this problem with the help of construction of the corresponding functional space and by establishing its equivalence to the Cauchy problem for some operator equation.

This problem was considered for the first time by Sobolev, in connection with the study of small oscillations of a rotating ideal fluid, in the particular case when $L = \Delta$ is the three-dimensional Laplace operator. Later similar problems were considered by R.A. Aleksandryan [1], S.A. Galpern [2] and others (see, e.g.,[3–5]).

1. Let Ω be a bounded domain in n-dimensional vector space R^n located in the half-space $x_n > 0$. We suppose that the boundary of the domain has the form

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$\partial\Omega = \Gamma_1 \cup \Gamma_0$, where $\Gamma_0 = \partial\Omega \cap \{x_n = 0\}$ is a domain in the hyperplane $\{x_n = 0\}$, $\Gamma_1 = \partial\Omega \backslash \Gamma_0$, and for the domain Ω the Sobolev embedding theorems are valid.

We consider the following initial-boundary value problem in the cylinder $Q = \Omega \times R^+$ for the degenerate pseudohyperbolic equation

$$\begin{cases} L\left(\dfrac{\partial^2 u}{\partial t^2}\right) + M(u) = 0, & (1) \\ u\big|_{t=0} = u^{(0)}(x),\ u_t\big|_{t=0} = u^{(1)}(x), & (2) \\ u\big|_{\Gamma^*} = 0, \quad t > 0, & (3) \end{cases}$$

where
$$Lu = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(b_{ij}(x,t) \frac{\partial u}{\partial x_j} \right) - \frac{\partial}{\partial x_n}\left(b_{nn}(x,t) \frac{\partial u}{\partial x_n} \right),$$
$$Mu = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t) \frac{\partial u}{\partial x_j} \right) - \frac{\partial}{\partial x_n}\left(a_{nn}(x,t) \frac{\partial u}{\partial x_n} \right).$$

We assume that the coefficients of the operators L and M are symmetric: $a_{ij}(x,t) = a_{ji}(x,t),\ b_{ij}(x,t) = b_{ji}(x,t)\ (i,j = 1,2,\ldots,n)$ continuous and bounded in $\overline{\Omega} \times R^+$, continuously differentiable with respect to the variables $x_1, x_2, \ldots, x_n$ in $Q = \Omega \times R^+$, there exist exponents $\alpha > \beta \geq 0$, such that the products $x_n^{-\beta} b_{nn}(x,t), | x_n^{-\alpha} a_{nn}(x,t) |$ are bounded from above and below by positive constants and for every $x \in \overline{\Omega}$ and every $t \geq 0$ the quadratic form $\Lambda_0(x,t,\hat{\xi}) = \sum_{i,j=1}^{n-1} b_{ij}(x,t)\xi_i\xi_j$ is positive-definite, where $\hat{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in R^{n-1}$.

Γ^* is a part of the boundary, which depending on the order of degeneracy β, represents either the whole boundary $\partial\Omega$ or coincides with the Γ_1.

2. In the space $L_2(\Omega)$ we define the operator L_β with the domain of definition $C_0^\infty(\Omega)$ by the formula

$$L_\beta(u) = -\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial}{\partial x_n}\left(x_n^\beta \frac{\partial u}{\partial x_n} \right).$$

It follows from the results of [5], that the operator L_β is symmetric and positive-definite. Define the Hilbert space H_{L_β} as the completion of the linear manifold $C_0^\infty(\Omega)$ in the metric generated by the following scalar product

$$(u,v)_{L_\beta} = \left(L_\beta(u), v\right)_0 = \int_\Omega \left[\sum_{i=1}^{n-1} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} + x_n^\beta \frac{\partial u}{\partial x_n} \cdot \frac{\partial v}{\partial x_n} \right] dx. \tag{4}$$

Let $T > 0$, $Q = \Omega \times (0,T)$ be a cylinder with the base Ω, $\Sigma = \Gamma \times (0,T)$ be the lateral boundary of the cylinder Q.

Definition. Twice differentiable in H_{L_β} trajectory $u(t)$ is called a weak or generalized solution of the problem (1)–(3), if $u_{|t=0}=u^{(0)}(x)$, $u_{t_{|t=0}}=u^{(1)}(x)$ and for every $v\in C_0^\infty(\Omega)$ and every $t>0$

$$\int_\Omega \frac{d^2u}{dt^2}\left[\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(b_{ij}(x,t)\frac{\partial v}{\partial x_j}\right)+\frac{\partial}{\partial x_n}\left(b_{nn}(x,t)\frac{\partial v}{\partial x_n}\right)\right]dx+$$

$$+\int_\Omega u\left[\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial v}{\partial x_j}\right)+\frac{\partial}{\partial x_n}\left(a_{nn}(x,t)\frac{\partial v}{\partial x_n}\right)\right]dx=0.$$

Theorem. For any initial values $u^{(0)}\in H_{L_\beta}$ and $u^{(1)}\in H_{L_\beta}$ there exists a unique generalized solution of the problem (1)–(3) in H_{L_β}. Where for the case $\beta<1$ we have $\Gamma^*=\partial\Omega$ and $\Gamma^*=\Gamma_1=\partial\Omega\setminus\Gamma_0$ for $\beta\ge1$.

Proof. Let $t_0\in[0,\infty)$ be a fixed number. In the space $L_2(\Omega)$ we define the operator $L(t_0)$ with the domain of definition $C_0^\infty(\Omega)$ by the formula

$$L(t_0)v=-\sum_{i,j=1}^{n-1}\frac{\partial u}{\partial x_i}\left(b_{ij}(x,t_0)\frac{\partial v}{\partial x_j}\right)-\frac{\partial}{\partial x_n}\left(b_{nn}(x,t_0)\frac{\partial v}{\partial x_n}\right).$$

Since the quadratic form $\Lambda(x,t_0,\overrightarrow{\xi})=\sum_{i,j=1}^{n-1}b_{ij}(x,t_0)\xi_i\xi_j+b_{nn}(x,t_0)\xi_n^2$ is positive-definite for every $x\in\overline{\Omega}\setminus\Gamma_0$ and

$$\Lambda(x,t_0,\overrightarrow{\xi})=\Lambda_0(x,t_0,\hat{\xi})+b_{nn}(x,t)\xi_n^2\ge\Lambda_0(x,t_0,\hat{\xi})\ge c\,|\,\xi\,|^2,$$

we conclude that the operator $L(t_0)$ is symmetric. It is easy to verify that the operator $L(t_0)$ is positive-definite in $L_2(\Omega)$. Moreover, since the quadratic form $\Lambda_0(x,t_0,\hat{\xi})$ is positive-definite, the product $x_n^{-\beta}b_{nn}(x,t_0)$ and the functions $b_{ij}(x,t_0)$, $i,j=1,2,\ldots,\ n-1$, are bounded, we deduce that there exist constants c_β and C_β, such that for every $v\in C_0^\infty(\Omega)$ we have

$$\begin{aligned}c_\beta(L_\beta(v),v)&=c_\beta\int_\Omega\left[\sum_{i=1}^{n-1}\left(\frac{\partial v}{\partial x_i}\right)^2+x_n^\beta\left(\frac{\partial v}{\partial x_n}\right)^2\right]dx\le\\ &\le\int_\Omega\left[\sum_{i=1}^{n-1}b_{ij}(x,t_0)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}+b_{nn}(x,t_0)\left(\frac{\partial v}{\partial x_n}\right)^2\right]dx\le C_\beta\left(L_\beta(v),v\right)_0.\end{aligned}\tag{5}$$

On the linear manifold $C_0^\infty(\Omega)$ we define the operator $A(t_0)$ by the formula

$$A(t_0):v\to\widehat{L}^{-1}(t_0)M(t_0)v,$$

where $\widehat{L}(t_0)$ is the Friedrich's extension of $L(t_0)$. It is easy to check that $A(t_0)$ is bounded in the space H_{L_β}. Indeed, let $v_1(x),v_2(x)\in C_0^\infty(\Omega)$ are arbitrary functions.

Then we have $\left(A(t_0)v_1,v_2\right)_{L(t_0)}=\left(v_1,A(t_0)v_2\right)_{L(t_0)}$, i.e. $A(t_0)$ is symmetric in the space $H_{L(t_0)}$. Moreover, there is a constant $c_1>0$ such that for every $v\in C_0^\infty(\Omega)$ we have

$$\left|\left(A(t_0)v,v\right)_{L(t_0)}\right|=\left|\int\limits_\Omega\left[\sum_{i,j=1}^{n-1}a_{ij}(x,t_0)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}+a_{nn}(x,t_0)\left(\frac{\partial v}{\partial x_n}\right)^2\right]dx\right|\le$$
$$\le\int\limits_\Omega\left[\sum_{i,j=1}^{n-1}|a_{ij}(x,t_0)|\cdot\left|\frac{\partial v}{\partial x_i}\right|\cdot\left|\frac{\partial v}{\partial x_j}\right|+|a_{nn}(x,t_0)|\cdot\left|\frac{\partial v}{\partial x_n}\right|^2\right]dx\le c_1\int\limits_\Omega\left[\sum_{i=1}^{n-1}\left(\frac{\partial v}{\partial x_i}\right)^2+x_n^\alpha\left(\frac{\partial v}{\partial x_n}\right)^2\right]dx. \tag{6}$$

Since $\alpha\ge\beta$, we obtain $x_n^\alpha\le c_1'x_n^\beta\le c_2'b_{nn}(x,t_0)$, where c_1' and c_2' are some positive constants. Consequently, from inequality (6) we conclude that there exists a constant $c_2>0$ such that

$$\left|\left(A(t_0)v,v\right)_{L(t_0)}\right|=\int\limits_\Omega\left[\sum_{i=1}^{n-1}\left(\frac{\partial v}{\partial x_i}\right)^2+x_n^\alpha\left(\frac{\partial v}{\partial x_n}\right)^2\right]dx\le$$
$$\le c_2\int\limits_\Omega\left[\sum_{i,j=1}^{n-1}b_{ij}(x,t_0)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}+b_{nn}(x,t_0)\left|\frac{\partial v}{\partial x_n}\right|^2\right]dx=c_2\left(L(t)v,v\right)_0=c_2\,\|v\|^2_{L(t_0)},$$

i.e., the operator $A(t_0)$ is bounded in the space $H_{A(t_0)}$. From the inequality (5) we deduce that $A(t_0)$ is bounded in the space H_{L_β}.

We extend the operator $A(t_0)$ by continuity from the linear manifold $C_0^\infty(\Omega)$ to the whole space H_{L_β}. The extension will be denoted by $\overline{A}(t_0)$. In the Hilbert space H_{L_β} we consider the following auxiliary Cauchy problem

$$\begin{cases}\dfrac{d^2u}{dt^2}=-\overline{A}(t_0)u,\\ u_{|t=0}=u^{(0)},u_{t|t=0}=u^{(1)}.\end{cases} \tag{7}$$

It is easy to see, that every solution of the problem (7) is a weak solution for the problem (1)–(3) and vice-versa. The boundedness of the operator $\overline{A}(t)$ in the space H_{L_β} implies that the problem (7) has a unique solution in H_{L_β}.

Thus, the Theorem is proved.

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