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Mechanics

A LOAD TRANSFER FROM AN ELASTIC INFINITE STRINGER TO AN ELASTIC SEMI-INFINITE PLATE

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A contact problem for a solid isotropic homogeneous elastic semi-infinite plate, strengthened by a solid homogeneous infinite elastic stringer (IES), has been investigated under assumption that the homogeneous IES is welded (glued) to the semi-infinite plate parallel to its boundary and is at a finite distance from that boundary. The contact pair (plate–stringer) was simultaneously deformed by uniformly distributed horizontal tensile stresses of constant intensity acting on the infinity of the plate, and by a concentrated force applied to the stringer and directed along the axis of stringer. The problem was formulated as a singular integral equation with kernel consisting of singular and regular parts. Then for solution of this equation the generalized Fourier integral transform was used, by means of which this integral equation was reduced to a functional equation with respect to the Fourier transform of unknown function. The closed solution of the contact problem was constructed in an integral form. The tangential contact forces and normal stresses arising in the IES were determined. Asymptotic formulas describing the behavior of stresses both near and far from the application point of concentrated force were obtained.

***Keywords*:** plate, contact, stringer, generalized Fourier integral transform, singular integral equation, functional equation, singularity, asymptotic.

1. Let an isotropic elastic solid sheet in the form of thin homogeneous semi-infinite plate of constant thickness h be strengthened along $y = a\,(a > 0)$ line on its upper surface by h_S thick and d_S wide solid isotropic homogeneous infinite elastic stringer (IES) of sufficiently small constant rectangular cross-section. The IES is assumed to be parallel to the plate boundary and be at a finite distance from it.

The aim of the present work is to determine the distribution intensity of tangential contact forces acting along the contact line of the IES with the elastic semi-infinite plate, and normal stresses arising in the IES when the contacting pair (plate–stringer) is deformed by uniformly distributed horizontal tensile stresses of constant intensity σ_0 acting on the infinity of the isotropic homogeneous elastic semi-infinite plate, and simultaneously by a concentrated force $P\delta(x)\delta(y-a)$, applied to the homogeneous IES and directed along the stringer axis.

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In the contact problem with respect to the IES under consideration we accepted a model of uniaxial strain state in combination with the model of contact along the line [1–7], i.e., it was assumed that the intensity of distribution of unknown tangential contact forces was concentrated along the midline of contact area, and for the elastic semi-infinite plate we accepted the model of generalized stress state, due to which it was deformed as a half-plane.

Now, as to obtaining the resolving equation of the contact problem in question, note that in the horizontal direction the IES is deformed (stretched or compressed) being in a uniaxial stress state. Then, the equilibrium differential equation for an element of IES as written in generalized functions on the bases of Hooke law with due regard for the aforesaid has the following form:

$$\frac{du_S(x;a)}{dx} = -\frac{1}{2E_S F_S}\int_{-\infty}^{\infty} \operatorname{sgn}(s-x)\tau(s)\,ds - \frac{P}{2E_S F_S}\operatorname{sgn}x + \frac{\sigma_0}{E} \quad (-\infty < x < \infty), \tag{1.1}$$

and the boundary conditions and the equilibrium condition for IES are:

$$\text{a)}\ \left.\frac{du_S(x;a)}{dx}\right|_{|x|\to\infty} = \frac{\sigma_0}{E}; \qquad \text{b)}\ \int_{-\infty}^{\infty}\tau(s)\,ds = P. \tag{1.2}$$

It is easy to see, that the normal stresses arising in the IES on $y=a$ line can be determined from (1.1) in the following form:

$$\sigma_x(x;a) = -\frac{1}{2F_S}\int_{-\infty}^{\infty}\operatorname{sgn}(s-x)\tau(s)\,ds - \frac{P}{2F_S}\operatorname{sgn}x + \frac{E_S}{E}\sigma_0 \quad (-\infty < x < \infty). \tag{1.3}$$

Here, in formulas (1.1) – (1.3) $u_S(x;a)$ are the horizontal displacements of IES on $y=a$ line; $\tau(x) = d_S\tau(x;a)$, where $\tau(x;a)$ are the tangential contact stresses on $y=a$ line; E_S and E are the Young modulus of the IES and the elastic semi-infinite plate respectively; $\operatorname{sgn}x$ is the well known unit sign function; P is the intensity of the concentrated force applied to the IES at $(0;a)$ point; $F_S = h_S d_S$ is the cross-sectional area of the IES.

From the other side, when on the $y=a$ line are acting tangential contact forces with intensity $\tau(x)$ and at the infinity of the elastic semi-infinite plate are acting uniformly distributed horizontal tensile stresses of constant intensity σ_0, for the horizontal deformation of the plate we have [2,3]:

$$hl\frac{du(x;a)}{dx} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{s-x} + K(s-x)\right]\tau(s)\,ds + \frac{hl}{E}\sigma_0 \quad (-\infty < x < \infty). \tag{1.4}$$

Here the following notations are introduced:

$$K(t) = \frac{d_1 t}{t^2+4a^2} - \frac{8a^2 t}{\left(t^2+4a^2\right)^2} - \frac{4a^2 d_2 t\left(t^2-12a^2\right)}{\left(t^2+4a^2\right)^3} \qquad (-\infty < t < \infty),$$

$$l = \frac{4E}{(1+\nu)(3-\nu)}, \quad d_1 = \frac{(1-\nu)^2+4}{(1+\nu)(3-\nu)}, \quad d_2 = \frac{1+\nu}{3-\nu}, \tag{1.5}$$

where $u(x;a)$ are the horizontal displacements of the points of elastic semi-infinite plate on $y=a$ line; ν is the Poisson ratio of the plate.

Note that on the contact line of the IES and the plate the following contact condition must be satisfied:

$$\frac{du_S(x;a)}{dx}=\frac{du(x;a)}{dx} \qquad (-\infty<x<\infty). \tag{1.6}$$

Now, making a comparison (1.1) and (1.4) with contact condition (1.6), we have the following singular integral equation with respect to unknown distribution intensity $\tau(x)$ of tangential contact forces, which is the basic unknown function of the contact problem under consideration with kernel consisting of singular and regular parts:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{s-x}+\frac{\lambda\pi}{2}\operatorname{sgn}(s-x)+K(s-x)\right]\tau(s)\,ds=-\frac{\lambda P}{2}\operatorname{sgn}x \qquad (-\infty<x<\infty), \tag{1.7}$$

where $\lambda=\dfrac{hl}{E_S F_S}$. Note, that when $s=x$ the integral with Cauchy kernel in the singular integral equation (1.7) is treated in the sense of its general value of Cauchy.

Thus, under the assumptions made the solution of contact problem in question is reduced to the solution of singular integral equation (1.7) under condition (1.2, b) with kernel consisting of singular and regular parts.

2. For solution of the singular integral equation (1.7) under condition (1.2, b) we first apply the generalized Fourier integral transform to the singular integral equation (1.7) and using the formula of convolution obtain after some transformations the following functional equation with respect to the Fourier transform of the distribution function of unknown tangential contact forces with intensity $\tau(x)$:

$$\left[\lambda+|\sigma|+K_1(|\sigma|)\right]\overline{\tau}(\sigma)=\lambda P \qquad (-\infty<\sigma<\infty), \tag{2.1}$$

where the function $K_1(|\sigma|)$ has the following form:

$$K_1(|\sigma|)=\left(d_1|\sigma|-2a\sigma^2+2d_2a^2|\sigma|^3\right)e^{-2a|\sigma|}. \tag{2.2}$$

It is easy to see that the equilibrium condition (1.2, b) for IES will assume the form:

$$\overline{\tau}(0)=P. \tag{2.3}$$

From functional equation (2.1) we obtain for $\overline{\tau}(\sigma)$ the following form:

$$\overline{\tau}(\sigma)=\frac{\lambda P}{\lambda+|\sigma|+K_1(|\sigma|)} \qquad (-\infty<\sigma<\infty). \tag{2.4}$$

Here $\overline{\tau}(\sigma)=F[\tau(x)]$ is the Fourier transform of $\tau(x)$ function; $F[\cdot]$ is the Fourier operator; σ is the spectral parameter of Fourier transform. Here it is noteworthy that as $\overline{\tau}(\sigma)$ function determined by (2.4) uniformly satisfies condition (2.3) and is even, hence, $\tau(x)$ is also even, as was to be expected. As was assumed above [3, 8]

$$
\begin{gathered}
F[1]=2\pi\delta(\sigma), \qquad F[\operatorname{sgn} t]=\frac{2i}{\sigma}, \\
F\left[\frac{1}{t}\right]=i\pi\operatorname{sgn}\sigma, \qquad F\left[\frac{t}{t^2+y^2}\right]=i\pi\operatorname{sgn}\sigma\cdot e^{-|\sigma y|}, \\
F\left[\frac{t}{(t^2+y^2)^2}\right]=\frac{i\pi\sigma}{2|y|}e^{-|\sigma y|}, \\
F\left[\frac{t}{(t^2+y^2)^3}\right]=\frac{i\pi\sigma}{8|y|^3}(1+|\sigma y|)e^{-|\sigma y|} \quad (-\infty<\sigma;\, t;\, y<\infty).
\end{gathered} \tag{2.5}
$$

For determination of the Fourier transform of $\sigma_x(x;a)$ function of normal stresses arising in the IES, we apply the generalized Fourier integral transform to (1.3) with due regard for (2.4) and obtain

$$
\bar{\sigma}_x(\sigma;a)=\frac{P\left(|\sigma|+K_1(|\sigma|)\right)}{F_S i\sigma\left(\lambda+|\sigma|+K_1(|\sigma|)\right)}+\frac{E_S}{E}2\pi\sigma_0\delta(\sigma) \quad (-\infty<\sigma<\infty), \tag{2.6}
$$

where $\bar{\sigma}_x(\sigma;a)=F\left[\sigma_x(x;a)\right]$ is the Fourier transform of $\sigma_x(x;a)$ function. Applying now the generalized Fourier integral inverse transform to (2.4) and (2.6), we finally obtain the unknown functions $\tau(x)$ and $\sigma_x(x;a)$ in the form

$$
\tau(x)=\frac{\lambda P}{\pi}\int_0^\infty\frac{\cos(\sigma x)}{\lambda+\sigma+K_1(\sigma)}d\sigma, \tag{2.7}
$$

$$
\sigma_x(x;a)=-\frac{P}{\pi F_S}\int_0^\infty\frac{\left(\sigma+K_1(\sigma)\right)\sin(\sigma x)}{\sigma\left(\lambda+\sigma+K_1(\sigma)\right)}d\sigma+\frac{E_S}{E}\sigma_0 \quad (-\infty<x<\infty). \tag{2.8}
$$

In the particular case, when the concentrated force is absent $(P=0)$, we obtain from (2.4) and (2.6), $\bar{\tau}(\sigma)=0$ and $\bar{\sigma}_x(\sigma;a)=\frac{E_S}{E}2\pi\sigma_0\delta(\sigma)$ $(-\infty<\sigma<\infty)$ respectively, whence it follows that $\tau(x)=0$ and $\sigma_x(x;a)=\frac{E_S}{E}\sigma_0$ $(-\infty<x<\infty)$. This particular case demonstrates that despite the fact, that the base is continuously deformed by stresses σ_0, there is no any load transfer from the IES to the semi-infinite plate.

Thus, we determined the distribution of the intensity of unknown tangential contact stresses along the contact line of homogeneous elastic semi-infinite plate with IES, that define the closed solution of the problem in an integral form, and unknown functions have forms (2.7) and (2.8).

3. Now investigate the behavior of functions $\tau(x)$ of the intensity distribution of tangential contact forces and $\sigma_x(x;a)$ of normal stresses arising in the homogeneous IES that characterize their behavior both near and far from the application point of concentrated force. First let us obtain asymptotic formulae for $\tau(x)$ function when $|x|\to\infty$. This aim in view it is easy to see that in case of $|\sigma|\to 0$ the following asymptotic formulae ensue from (2.4):

$$\bar{\tau}(\sigma)=\left[1-a_1|\sigma|+a_2\sigma^2-a_3|\sigma|^3+a_4\sigma^4-a_5|\sigma|^5+a_6\sigma^6\right]P+O\left(|\sigma|^7\right), \quad (3.1)$$

where the following notations are introduced:

$$a_1=8c;\ a_2=16c(a+4c);\ a_3=8c\left[(3+\nu)a^2+32ac+64c^2\right],$$

$$a_4=\frac{4c}{3}\left[\left((5+\nu)^2-8\right)a^3+96(5+\nu)a^2c+2304ac^2+3072c^3\right],$$

$$a_5=\frac{2c}{3}\left[\left(3(3+\nu)^2-4\right)a^4+32\left((11+\nu)^2-68\right)a^3c+2304(7+\nu)a^2c^2+\right.$$
$$\left.+49152ac^3+49152c^4\right], \quad (3.2)$$

$$a_6=\frac{8c}{45}\left[9(3+\nu)\left(\frac{5}{3}+\nu\right)a^5+780(5+\nu)\left(\frac{29}{13}+\nu\right)a^4c+1440\left((17+\nu)^2-184\right)a^3c^2+\right.$$
$$\left.+92160(9+\nu)a^2c^3+2867200ac^4+1474560c^5\right];$$

$$c=\frac{1}{\lambda(1+\nu)(3-\nu)}.$$

If we apply the inverse generalized Fourier integral transform to (3.1) and take into account that [3, 8]

$$F^{-1}\left[\sigma^{2n}\right]=(-1)^n\delta^{(2n)}(x);\ F^{-1}\left[|\sigma|^{2n+1}\right]=(-1)^{n+1}\frac{\Gamma(2n+2)}{\pi\cdot x^{2n+2}}\ (n=0,1,2,...,), \quad (3.3)$$

where $F^{-1}[\cdot]$ is the Fourier inverse operator; $\Gamma(x)$ is the gamma-function; $\delta^{(n)}(x)$ is the Dirac`s delta-function's $\delta(x)$ n-th derivative, we obtain for $\tau(x)$ function the following asymptotic representation when $|x|\to\infty$:

$$\tau(x)=\left[\frac{a_1}{x^2}-\frac{6a_3}{x^4}+\frac{120a_5}{x^6}\right]\frac{P}{\pi}+O\left(\frac{1}{x^8}\right). \quad (3.4)$$

For obtaining the asymptotic formulae for $\sigma_x(x;a)$ function in case of $|x|\to\infty$, note that when $|\sigma|\to 0$ one can obtain the following asymptotic representation for $\bar{\sigma}_x(\sigma;a)$ function from (2.6):

$$\bar{\sigma}_x(\sigma;a)=-\frac{iP}{F_S}\left[a_1\operatorname{sgn}\sigma-a_2|\sigma|\operatorname{sgn}\sigma+a_3\sigma^2\operatorname{sgn}\sigma-a_4|\sigma|^3\operatorname{sgn}\sigma+\right.$$
$$\left.+a_5\sigma^4\operatorname{sgn}\sigma-a_6|\sigma|^5\operatorname{sgn}\sigma\right]+\frac{E_S}{E}2\pi\sigma_0\delta(\sigma)+O\left(\sigma^6\operatorname{sgn}\sigma\right). \quad (3.5)$$

If we apply the inverse generalized integral Fourier transform to (3.5), then keeping in mind that [3, 8]:

$$F^{-1}\left[\sigma^n\operatorname{sgn}\sigma\right]=\frac{\Gamma(n+1)}{\pi}\cdot\frac{1}{(ix)^{n+1}},$$
$$F^{-1}\left[|\sigma|^n\operatorname{sgn}\sigma\right]=\frac{\Gamma(n+1)}{i\pi}\cos\frac{\pi n}{2}\cdot\frac{\operatorname{sgn}x}{|x|^{n+1}}, \quad (n=0,1,2,...), \quad (3.6)$$

we obtain for $\sigma_x(x;a)$ function the following asymptotic formulae when $|x|\to\infty$:

$$\sigma_x(x;a)=-\left[\frac{a_1}{x}-\frac{2a_3}{x^3}+\frac{24a_5}{x^5}\right]\frac{P}{\pi F_S}+\frac{E_S}{E}\sigma_0+O\left(\frac{1}{x^7}\right). \qquad (3.7)$$

From obtained asymptotic formulas follows, that when $|x|\to\infty$ tangential contact stresses have an order of $O(x^{-2})$, and normal stresses have an order of $O(x^{-1})$.

Now let us obtain asymptotic formulas for functions $\tau(x)$ and $\sigma_x(x;a)$ when $|x|\to 0$. First represent the function $\overline{\tau}(\sigma)$ $(-\infty<\sigma<\infty)$, determined by (2.4), as follows:

$$\overline{\tau}(\sigma)=\frac{\lambda P}{\lambda+|\sigma|}-\frac{\lambda P K_1(|\sigma|)}{(\lambda+|\sigma|)(\lambda+|\sigma|+K_1(|\sigma|))} \qquad (-\infty<\sigma<\infty). \qquad (3.8)$$

As the following asymptotic representation holds when $|\sigma|\to\infty$

$$\frac{\lambda}{\lambda+|\sigma|}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{\lambda}{|\sigma|}\right)^{n+1}, \qquad (3.9)$$

then by applying inverse generalized Fourier integral transform to (3.8) and taking into account that [3, 8]

$$F^{-1}\left[|\sigma|^{-2n-1}\right]=\frac{(-1)^n x^{2n}}{\pi(2n)!}\left[\psi(2n+1)+\ln\frac{1}{|x|}\right];$$
$$F^{-1}\left[\sigma^{-2n-2}\right]=(-1)^{n+1}\frac{|x|^{2n+1}}{2(2n+1)!};\quad (n=0,1,2,...), \qquad (3.10)$$

we obtain the following asymptotic representation for $\tau(x)$ function when $|x|\to 0$

$$\tau(x)=\frac{\lambda P}{\pi}\left(\psi(1)-A_0+\ln\frac{1}{|\lambda x|}+\frac{\pi|\lambda x|}{2}\right)+$$
$$+\frac{\lambda P}{\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(\lambda x)^{2n}}{(2n)!}\left(\psi(2n+1)+\ln\frac{1}{|\lambda x|}\right)+\frac{\pi|\lambda x|^{2n+1}}{2(2n+1)!}-A_n\frac{x^{2n}}{(2n)!}\right], \qquad (3.11)$$

where the unknown coefficients A_n $(n=0,1,2,...)$ are

$$A_n=\int_0^{\infty}\frac{K_1(\sigma)\sigma^{2n}}{(\lambda+\sigma)(\lambda+\sigma+K_1(\sigma))}d\sigma \qquad (3.12)$$

and $\psi(x)$ is the well known psi-function, $\psi(1)=-\gamma=-0,5772...$ being the Euler constant. It follows from asymptotic formulae (3.11) that when $|x|\to 0$, the function $\tau(x)$ has a logarithmic singularity due to the presence of concentrated force P. To obtain asymptotic formulae for $\sigma_x(x;a)$ function when $|x|\to 0$, (1.3) is represented in the following form:

$$\sigma_x(x;a)=\frac{1}{F_S}\int_0^x\tau(s)ds-\frac{P}{2F_S}\operatorname{sgn}x+\frac{E_S}{E}\sigma_0. \qquad (3.13)$$

Then, taking into account (3.11), we obtain from (3.13) that in case of $|x| \to 0$

$$\sigma_x(x;a) = \frac{P}{\pi F_S}\left[(\lambda x)\left(\psi(2) - A_0 + \ln\frac{1}{|\lambda x|} + \frac{\pi|\lambda x|}{4}\right)\right] - \frac{P}{2F_S}\operatorname{sgn} x + \frac{E_S}{E}\sigma_0 +$$
$$+\frac{P}{\pi F_S}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(\lambda x)^{2n+1}}{(2n+1)!}\left(\psi(2n+2) + \ln\frac{1}{|\lambda x|} + \frac{\pi|\lambda x|}{2(2n+2)}\right) - A_n\frac{\lambda x^{2n+1}}{(2n+1)!}\right]. \quad (3.14)$$

Note at the end that the series (3.11) and (3.14) are convergent for any x $(-\infty < x < \infty)$.

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