PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2012, № 3, p. 3–8

Mathematics

MEAN DISTANCE BETWEEN TWO POINTS IN A DOMAIN

N. G. AHARONYAN*

Chair of Probability Theory and Mathematical Statistics YSU, Armenia

Let D be a bounded convex domain in the Euclidean plane and we choose uniformly and independently two points in D. How large is the mean distance m(D) between these two points? Up to now, there were known explicit expressions for m(D) only in three cases, when D is a disc, an equilateral triangle and a rectangle. In the present paper a formula for calculation of mean distance m(D) by means of the chord length density function of D is obtained. This formula allows to find m(D) for those domains D, for which the chord length distribution is known. In particular, using this formula, we derive explicit forms of m(D) for a disc, a regular triangle, a rectangle, a regular hexagon and a rhombus.

***Keywords*:** chord length distribution function, mean distance, convex domain geometry.

Introduction. Let D be a bounded convex domain in the Euclidean plane $\mathbb{R}^2$ (with area $\|\mathrm{D}\|$ and perimeter $|\mathrm{D}|$), and we choose uniformly and independently two points P_1 and P_2 from D. How large is the mean distance $m(\mathrm{D})$ between these two points? In other words, we want to calculate the following quantity [1]:

$$m(\mathrm{D}) = 1/\|\mathrm{D}\|^2 \int_{\mathrm{D}}\int_{\mathrm{D}} \rho(P_1,P_2)dP_1dP_2, \tag{1.1}$$

where $\rho(P_1,P_2)$ is the Euclidean distance between P_1 and P_2, dP_i $(i=1,2)$ is the two dimensional Lebesgue measure. Up to now, there were known explicit expressions for $m(\mathrm{D})$ only in three cases when D is the disc K_r of a radius r [2]:

$$m(K_r) = 128r/45\pi \tag{1.2}$$

for an equilateral triangle Δ_a of side a:

$$m(\Delta_a) = a(1/5 + 3\ln/20), \tag{1.3}$$

and for a rectangle $R_{a,b}$ with sides $a \le b$ [3]:

$$m(R_{a,b}) = \frac{1}{15}\left[\frac{a^3}{b^2} + \frac{b^3}{a^2} + \sqrt{a^2+b^2}\left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2}\right) + \frac{5}{2}\left(\frac{b^2}{a}\ln\frac{a+\sqrt{a^2+b^2}}{b} + \frac{a^2}{b}\ln\frac{b+\sqrt{a^2+b^2}}{a}\right)\right]. \tag{1.4}$$

* E-mail: narine78@ysu.am

In the present paper a formula for calculation of mean distance $m(\mathrm{D})$ by means of the chord length density function of D is obtained, which allows to find an explicit form of mean distance $m(\mathrm{D})$ for those domains D, for which the chord length distribution is known. In particular, using this formula, we derive explicit forms of $m(\mathrm{D})$ for a disc, a regular triangle, a rectangle, a regular hexagon and a rhombus.

2. The Main Formula. In this section we obtain a formula for calculation $m(\mathrm{D})$ in terms of chord length density function for domain D.

Let $\chi(g) = g \cap \mathrm{D},\ g \in [\mathrm{D}]$, is a chord in D, where $[\mathrm{D}] = \{g \in \mathbb{G} : g \cap \mathrm{D} \neq \varnothing\}$ is the set of lines intersecting domain D ($\mathbb{G}$ is the space of all undirected lines in the plane $\mathbb{R}^2$). It is well known the following formula between $m(\mathrm{D})$ and the integral $I_4 = \int\limits_{[\mathrm{D}]} \chi^4(g)\, dg$:

$$I_4 = 6 \,\|\mathrm{D}\|^2\, m(\mathrm{D}), \tag{2.1}$$

where dg is the locally finite measure in the space $\mathbb{G}$, which is invariant with respect to the group of all Euclidean motions in the plane [1, 4].

By definition, $F_{\mathrm{D}}(y) = 1/|\partial \mathrm{D}| \int\limits_{\chi(g)\le y} dg$ is the chord length distribution function for D [5, 6].

To transform the formula for $m(\mathrm{D})$, we write

$$I_4 = |\partial \mathrm{D}| \int\limits_0^{+\infty} x^4 f_{\mathrm{D}}(x)dx, \tag{2.2}$$

where $f_{\mathrm{D}}(x)$ is the chord length density function of D (that is $f_{\mathrm{D}}(x) = F'_{\mathrm{D}}(x)$ is the first derivative of the distribution function). Finally, from (2.2) and (2.1), we get

$$m(\mathrm{D}) = |\partial \mathrm{D}| / 6 \,\|\mathrm{D}\|^2 \int\limits_0^{+\infty} x^4 f_{\mathrm{D}}(x)dx, \tag{2.3}$$

Therefore, if we know the explicit form of $f_{\mathrm{D}}(x)$ for domain D, using (2.3) we can calculate $m(\mathrm{D})$. Further we demonstrate the efficiency of formula (2.3).

3. The Case of a Disc. Consider a disc of radius r, $\mathrm{D} = K_r$. In this case the chord length density function has the following form [5]:

$$f_{K_r}(x) = \begin{cases} 0, & \text{if } x \notin (0, 2r), \\ x/(4r^2\sqrt{1 - x^2/4r^2}\,), & \text{if } x \in (0, 2r). \end{cases}$$

Therefore, we get $m(K_r) = 1/6\pi r^4 \int\limits_0^{2r} x^5 / \sqrt{4r^2 - x^2}\, dx = 128r/45\pi$, i.e. (1.2) [1, 5, 7].

4. The Case of a Regular Triangle. The chord length density function for a regular triangle Δ_a with side a has the following form [5, 8, 9]:

$$f_{\Delta_a}(x) = \begin{cases} 0, \quad \text{if } x \notin [0, a]; & \dfrac{1}{2a} + \dfrac{\pi}{3a\sqrt{3}}, \quad \text{if } x \in \left(0, \dfrac{a\sqrt{3}}{2}\right]; \\ \dfrac{1}{2a} - \dfrac{\sqrt{4x^2 - 3a^2}}{2x^2} + \dfrac{2}{a\sqrt{3}} \arcsin \dfrac{a\sqrt{3}}{2x} - \dfrac{2\pi}{3a\sqrt{3}}, & \text{if } x \in \left(\dfrac{a\sqrt{3}}{2}, a\right]. \end{cases} \tag{4.1}$$

Substituting the values of $f_{\Delta_a}(x)$ from (4.1) into (2.3), we obtain

$$m(\Delta_a)=\frac{8}{3a^3}\left[\frac{1}{2a}\int_0^a x^4dx+\frac{\pi}{3a\sqrt{3}}\int_0^{\sqrt{3}a/2}x^4dx-\frac{2\pi}{3a\sqrt{3}}\int_{\sqrt{3}a/2}^{a}x^4dx-\right.$$
$$\left.-\frac{1}{2}\int_{\sqrt{3}a/2}^{a}x^2\sqrt{4x^2-3a^2}\,dx+\frac{2}{a\sqrt{3}}\int_{\sqrt{3}a/2}^{a}x^4\arcsin\frac{a\sqrt{3}}{2x}dx\right]. \tag{4.2}$$

Using some standard formulas we obtain $m(\Delta_a)=a(1/5+3\ln 3/20)$ (1.3) [1–3, 5]).

5. The Case of a Rectangle. It is well known that the chord length density function for a rectangle $R_{a,b}$ with sides $a\le b$ has the following form [5–7, 9, 10]:

$$f_{R_{a,b}}(x)=\begin{cases}0, \text{ if } x\notin[0,\sqrt{a^2+b^2}); & x/(a+b), \text{ if } x\in(0,a];\\ a^2b/(a+b)x^2\sqrt{x^2-a^2}, & \text{if } x\in(a,b];\\ \dfrac{ab}{(a+b)x^2}\left(\dfrac{a}{\sqrt{x^2-a^2}}+\dfrac{b}{\sqrt{x^2-b^2}}\right)-\dfrac{1}{a+b}, & \text{if } x\in(b,\sqrt{a^2+b^2}].\end{cases} \tag{5.1}$$

Substituting the values of $f_{R_{a,b}}(x)$ from (5.1) into (2.3), we obtain

$$m(R_{a,b})=\frac{1}{3a^2b^2}\left[\frac{a^5}{5}+a^2b\int_a^{\sqrt{a^2+b^2}}\frac{x^2}{\sqrt{x^2-a^2}}dx+\right.$$
$$\left.+ab^2\int_b^{\sqrt{a^2+b^2}}\frac{x^2}{\sqrt{x^2-b^2}}dx+\frac{b^5}{5}-\frac{(a^2+b^2)^2\sqrt{a^2+b^2}}{5}\right]. \tag{5.2}$$

After some calculations we obtain (1.4) [1, 2]. In the case $a=b$ we get the mean distance for a square with side a

$$m(\square_a)=a/15(2+\sqrt{2}+5\ln(1+\sqrt{2})). \tag{5.3}$$

6. The Case of a Regular Hexagon. For a regular hexagon H_a with side a we have: $|H_a|=6a$, $\|H_a\|=3\sqrt{3}a^2/2$. The density function has the following form (see [6, 8–10]):

$$f_{H_a}(y)=\begin{cases}0, \text{ if } y\notin(0,2a]; & \dfrac{1}{a}\left(\dfrac{1}{2}-\dfrac{\pi}{6\sqrt{3}}\right), \text{ if } y\in(0,a];\\ \dfrac{\pi}{2a\sqrt{3}}-\dfrac{2}{a\sqrt{3}}\arcsin\dfrac{a\sqrt{3}}{2y}+\dfrac{\sqrt{4y^2-3a^2}}{2y^2}, & \text{if } y\in(a,a\sqrt{3}];\\ \dfrac{\pi}{6a\sqrt{3}}-\dfrac{1}{2a}-\dfrac{1}{a\sqrt{3}}\arccos\dfrac{a\sqrt{3}}{y}+\dfrac{6a^2-y^2}{y^2\sqrt{y^2-3a^2}}, & \text{if } y\in(a\sqrt{3},2a].\end{cases} \tag{6.1}$$

Therefore, $m(H_a)=\frac{4}{27a^3}\int_0^\infty x^4 f_{H_a}(x)dx$. Substituting (6.1) form, we get

$m(H_a)=\frac{4}{27a^3}[I_1+I_2+I_3]$, where $I_1=\frac{1}{a}\int_0^a\left(\frac{1}{2}-\frac{\pi}{6\sqrt{3}}\right)x^4dx=a^4\left(\frac{1}{10}-\frac{\pi}{30\sqrt{3}}\right)$;

$$I_2=\frac{\pi}{2a\sqrt{3}}\int_a^{a\sqrt{3}}x^4dx-\frac{2}{a\sqrt{3}}\int_a^{a\sqrt{3}}x^4\arcsin\frac{a\sqrt{3}}{2x}dx+\int_a^{a\sqrt{3}}\frac{x^2\sqrt{4x^2-3a^2}}{2}dx;$$

$$I_3=\int_{a\sqrt{3}}^{2a}\left(\frac{\pi}{6a\sqrt{3}}-\frac{1}{2a}\right)x^4dx-\frac{1}{a\sqrt{3}}\int_{a\sqrt{3}}^{2a}x^4\arccos\frac{a\sqrt{3}}{x}dx+\int_{a\sqrt{3}}^{2a}\frac{6a^2-x^2}{\sqrt{x^2-3a^2}}x^2dx.$$

Calculating I_2 and I_3, we obtain finally:

$$m(H_a)=a\left(\frac{4\pi}{15}-\frac{227}{2160}+\frac{478\pi}{3645\sqrt{3}}+\frac{113\sqrt{3}}{240}-\frac{3}{160}\ln\frac{2+\sqrt{3}}{\sqrt{3}}+\frac{14}{15}\ln\sqrt{3}\right).$$

7. The Case of Rhombus. In terms of elementary functions the chord length distribution function for rhombus $R(a,\gamma)$ for $\pi/3<\gamma\le\pi/2$ has the form [10]:

$$f_{R(a,\gamma)}(x)=\begin{cases}0,\ \text{if } x\notin[0,2a\cos\gamma/2]; \qquad (1+(\pi/2-\gamma)\cot\gamma)/2a,\ \text{if } x\in[0,a\sin\gamma);\\ \dfrac{1}{2a}\left(1-\left(\dfrac{\pi}{2}+\gamma-2\arcsin\dfrac{a\sin\gamma}{x}\right)\cot\gamma\right)+\dfrac{a^2\sin^2\gamma+x^2\cos\gamma}{x^2\sqrt{x^2-a^2\sin^2\gamma}},\ \text{if } x\in[a\sin\gamma,a);\\ -\dfrac{1}{2a}\left(1+\left(\dfrac{\pi}{2}-\gamma\right)\cot\gamma\right)+\dfrac{a^2\sin^2\gamma}{x^2\sqrt{x^2-a^2\sin^2\gamma}},\ \text{if } x\in\left[a,2a\sin\dfrac{\gamma}{2}\right);\\ \dfrac{-1+\left(\gamma-2\arcsin\dfrac{a\sin\gamma}{x}\right)\cot\gamma}{4a}+\dfrac{a^2\sin^2\gamma+x^2\cos\gamma}{2x^2\sqrt{x^2-a^2\sin^2\gamma}},\ \text{if } x\in\left[2a\sin\dfrac{\gamma}{2},2a\cos\dfrac{\gamma}{2}\right).\end{cases}$$

Since $|R(a,\gamma)|=4a$, $||R(a,\gamma)||=a^2\sin\gamma$, using (2.3) we obtain

$$m(R(a,\gamma))=\frac{2}{3a^3\sin^2\gamma}\int_0^\infty x^4f_{R(a,\gamma)}(x)dx.$$

We denote by $m_1(R(a,\gamma))=\int_0^{a\sin\gamma}\frac{x^4}{2a}\left[1+\left(\frac{\pi}{2}-\gamma\right)\cot\gamma\right]dx$,

$$m_2(R(a,\gamma))=\int_{a\sin\gamma}^{a}\frac{x^4}{2a}\left[\left(1-\left(\frac{\pi}{2}+\gamma-2\arcsin\frac{a\sin\gamma}{x}\right)\cot\gamma\right)\right]dx$$

and $$m_3(R(a,\gamma))=\int_{a\sin\gamma}^{a}\frac{a^2x^2\sin^2\gamma+x^4\cos\gamma}{x^2\sqrt{x^2-a^2\sin^2\gamma}}dx.$$

We have $m_2(R(a,\gamma))=\dfrac{a^4}{10}-\dfrac{a^4\sin^5\gamma}{10}-\dfrac{a^4(\pi+2\gamma)\cot\gamma}{20}+\dfrac{a^4(\pi+2\gamma)\cos\gamma\sin^4\gamma}{20}+\dfrac{\cot\gamma}{a}G_1$,

where $G_1=a^5\sin^5\gamma\int_{\sin\gamma}^{1}(1/x^6)\arcsin x\,dx$. Calculating G_1 and $m^3(R(a,\gamma))$, we get

$$G_1=-a^5\sin^5\gamma\left[\frac{\pi}{10}-\frac{\gamma}{5\sin^5\gamma}-\frac{1}{5}\left[\frac{\cos\gamma}{4\sin^4\gamma}+\frac{3\cos\gamma}{8\sin^2\gamma}+\frac{3}{8}\ln\frac{1+\cos\gamma}{\sin\gamma}\right]\right]\ \text{and}$$

$$m_3(R(a,\gamma))=a^4\left[\frac{\sin^2\gamma\cos\gamma}{2}+\frac{\sin^4\gamma}{2}(1+\frac{3}{4}\cos\gamma)\ln\frac{1+\cos\gamma}{\sin\gamma}+\frac{\cos^2\gamma}{4}+\frac{3}{8}\sin^2\gamma\cos^2\gamma\right].$$

Further, we have

$$m_4(R(a,\gamma))=\left[\frac{a^4}{10}-\frac{16a^4\sin^5(\gamma/2)}{5}\right]\left[1+(\frac{\pi}{2}-\gamma)\cot\gamma\right]+M_1a^2\sin^2\gamma$$

$$m_5(R(a,\gamma))=\frac{8}{5}\left[a^4\sin^5(\gamma/2)-a^4\cos^5(\gamma/2)+\gamma a^4\cot\gamma\cos^5(\gamma/2)-\gamma a^4\cot\gamma\sin^5(\gamma/2)\right]-$$
$$-K_1\cot\gamma/2a,$$

where $M_1=\int\limits_a^{2a\sin(\gamma/2)}\frac{x^2}{\sqrt{x^2-a^2\sin^2\gamma}}dx$ and $K_1=\int\limits_{2a\sin(\gamma/2)}^{2a\cos(\gamma/2)}x^4\arcsin\frac{a\sin\gamma}{x}dx$,

and at least $m_6(R(a,\gamma))=\frac{1}{2}\int\limits_{2a\sin(\gamma/2)}^{2a\cos(\gamma/2)}\frac{a^2x^2\sin^2\gamma+x^4\cos\gamma}{\sqrt{x^2-a^2\sin^2\gamma}}dx$.

Using a standard integral, we obtain

$$M_1=a^2\left(2\sin^3\frac{\gamma}{2}-\frac{\cos\gamma}{2}\right)+\frac{a^2}{2}\sin^2\gamma\ln\frac{2\sin\frac{\gamma}{2}(1+\sin\frac{\gamma}{2})}{1+\cos\gamma},$$ besides we get

$$K_1=-a^5\sin^5\gamma\left[-\frac{\gamma}{10\sin^5(\gamma/2)}+\frac{\pi-\gamma}{10\cos^5(\gamma/2)}-\frac{\cos(\gamma/2)}{20\sin^4(\gamma/2)}+\frac{\sin(\gamma/2)}{20\cos^4(\gamma/2)}-\right.$$
$$\left.-\frac{3\cos(\gamma/2)}{40\sin^2(\gamma/2)}+\frac{3\sin(\gamma/2)}{40\cos^2(\gamma/2)}-\frac{3}{40}\ln\frac{\cos(\gamma/2)+\cos^2(\gamma/2)}{\sin(\gamma/2)+\sin^2(\gamma/2)}\right].$$

Since $m(R(a,\gamma))=\frac{2}{3a^3\sin^2\gamma}\sum\limits_{i=1}^{6}m_i(R(a,\gamma))$, finally we have

$$m(R(a,\gamma))=\frac{2a}{3\sin^2\gamma}\left[\frac{1}{5}+\frac{\sin\gamma\cos\gamma}{20}+\frac{3}{40}\sin^3\gamma\cos\gamma+\frac{1}{20}\sin^4\gamma(10+9\cos\gamma)\ln\frac{1+\cos\gamma}{\sin\gamma}+\frac{\cos^2\gamma}{4}+\right.$$
$$+\frac{3}{8}\sin^2\gamma\cos^2\gamma-\frac{8}{5}\sin^5\frac{\gamma}{2}+\frac{8\pi}{5}\sin^5\frac{\gamma}{2}\cot\gamma+\sin^2\gamma\sin^3\frac{\gamma}{2}+\frac{\sin^4\gamma}{2}\ln\frac{2\sin\frac{\gamma}{2}\left(1+\sin\frac{\gamma}{2}\right)}{1+\cos\gamma}-$$
$$-\frac{8}{5}\cos^5\frac{\gamma}{2}+\frac{8}{5}\gamma\cot\gamma\left(\cos^5\frac{\gamma}{2}-\sin^5\frac{\gamma}{2}\right)+\frac{\sin^4\gamma\cos\gamma}{2}\left[-\frac{\gamma}{10\sin^5(\gamma/2)}+\frac{\pi-\gamma}{10\cos^5(\gamma/2)}-\right.$$
$$\left.-\frac{\cos\frac{\gamma}{2}}{20\sin^4\frac{\gamma}{2}}+\frac{\sin\frac{\gamma}{2}}{20\cos^4\frac{\gamma}{2}}-\frac{3\cos\frac{\gamma}{2}}{40\sin^2\frac{\gamma}{2}}+\frac{3\sin\frac{\gamma}{2}}{40\cos^2\frac{\gamma}{2}}\right]+\sin^2\gamma\cos^3\frac{\gamma}{2}+2\cos\gamma\left(\cos^5\frac{\gamma}{2}-\sin^5\frac{\gamma}{2}\right)+$$
$$\left.+\frac{3}{4}\sin^2\gamma\cos^3\frac{\gamma}{2}\cos\gamma-\frac{3}{4}\sin^2\gamma\sin^3\frac{\gamma}{2}\cos\gamma+\frac{1}{80}\sin^4\gamma(20+3\cos\gamma)\ln\frac{\cos\frac{\gamma}{2}+\cos^2\frac{\gamma}{2}}{\sin\frac{\gamma}{2}+\sin^2\frac{\gamma}{2}}\right].$$

Therefore, $m(R(a,\gamma))=C(\gamma)a$, were $C(\gamma)$ is a constant depending only on the angel $\gamma\in[\pi/3,\pi/2]$.

In the case $\gamma = \pi/2$, $C(\pi/2)$ coincides with the rectangle (5.3)

$$C(\pi/2) = 1/15\left[2+\sqrt{2}+5\ln(1+\sqrt{2})\right].$$

Similarly we can consider the case $\gamma \in [0, \pi/3)$.

8. Conclusion. We have calculated explicit values for *m*(D) for some particular cases, for which there exist explicit form of the chord length density function. We would like to stress that in [8] there exist an explicit form for the chord length distribution function for any regular polygon. In particular, for a regular pentagon the explicit form for chord length distribution you can find in [5]. Therefore, using the result of [8], we can calculate *m*(D) for any regular polygon.

Received 18.06.2012

R E F E R E N C E S

1. **Santalo L.A.** Integral Geometry and Geometric Probability. M.: Nauka, 1983.
2. **Schnider R. and Weil W.** Stochastic and Integral Geometry. Berlin, Heidelberg: Springer Verlag, 2008.
3. **Sulanke R.** Die Verteilung der Sehnenl Angen an Ebenen und *r* Aumlichen Figuren. // Math. Nachr., 1961, v. 23, p. 51–74.
4. **Gille W.** The Chord Length Distribution of Parallelepipeds with Their Limiting Cases. // Exp. Techn. Phys., 1988, v. 36, p. 197–208.
5. **Aharonyan N. and Ohanyan V.** Chord Length Distribution Functions for Polygons. // Journal of Contemporary Mathem. Analysis (Armenian Academy of Sciences), 2005, v. 40, № 4, p. 43–56.
6. **Harutyunyan H.S.** Chord Length Distribution Functions for Regular Hexagon. // Uchenie Zapiski EGU, 2007, № 1, p. 17–24 (in Russian).
7. **Stoyan D. and Stoyan H.** Fractals, Random Shapes and Point Fields. Chichester, New York, Brisbane, Toronto, Singapore: John Wiley & Sons, 1994.
8. **Harutyunyan H.S. and Ohanyan V.K.** Chord Length Distribution Function for Regular Polygons. // Advances in Applied Probability, 2009, v. 41, p. 358–366.
9. **Aharonyan N.G. and Ohanyan V.K.** Tomography of Bounded Convex Domains. // Sutra: International Journal of Mathematical Science Education, 2009, v. 2, № 1, p. 1–12.
10. **Harutyunyan H.S. and Ohanyan V.K.** Chord Length Distribution Cunction for Convex Polygons. // Sutra: International Journal of Mathem. Science Education, 2011, v. 4, № 2, p. 1–15.
11. **Burgstaller B. and Pillichshammer F.** The Average Distance Between Two Points. // Bull. Aust. Math. Soc., 2009, v. 80, p. 353–359.
12. **Dunbar S.R.** The Average Distance Between Points in Geometric Gures. // College Math. J., 1997, v. 28, p. 187–197.