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*Mathematics*

# ON THE REMINDER OF A TAYLOR–MACLAURIN TYPE GENERAL FORMULA

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We continue the study started in [1] regarding the Taylor–Maclaurin representation of functions. In the present paper we obtain a more convenient representation for the function $R_n(x)$, the residual term of the generalized Taylor–Maclaurin formula.

***Keywords*:** Weil operators, Taylor–Maclaurin type formulae.

**§ 1. Preliminaries.** It is known that the function

$$E_\rho(z;\mu)=\sum_{n=0}^{\infty}(z^n/\Gamma(\mu+n\rho^{-1}))(\rho>0) \tag{1.1}$$

is an entire function of order $\rho$ and type 1 for any value of the parameter $\mu$ [2].

For any $\mu>0$, $\alpha>0$ the following formula holds

$$1/\Gamma(\alpha)\int_0^z(z-t)^{\alpha-1}E_\rho(\lambda t^{1/\rho};\mu)t^{\mu-1}dt=z^{\mu+\alpha-1}E_\rho(\lambda z^{1/\rho};\mu+\alpha), \tag{1.2}$$

where $\lambda$ is complex parameter, and the integration is taken along the line segment connecting points 0 and $z$ (see [2], chap. III, (1.16)).

It can be shown also that for any $\alpha>0$, $\beta>0$ the following formula holds:

$$\int_0^l x^{\alpha-1}E_\rho(\lambda x^{1/\rho};\alpha)(l-x)^{\beta-1}E_\rho(\lambda^*(l-x)^{1/\rho};\beta)dx= \\ =l^{\alpha+\beta-1}\lambda E_\rho(l^{1/\rho}\lambda;\alpha+\beta)-\lambda^*E_\rho(l^{1/\rho}\lambda^*;\alpha+\beta)/(\lambda-\lambda^*), \tag{1.3}$$

where $\lambda$, $\lambda^*$ ($\lambda\neq\lambda^*$) are arbitrary complex parameters (see [2], chap. III, (1.21)). From (1.3), using the well-known formula $E_p(z;2/\rho)=1/z\{E_p(z;\ 1/\rho)-1/\Gamma(\rho^{-1})\}$, for the particular case of $\alpha=\beta=1/\rho$ we can deduce that

$$\int_0^x E_\rho(\lambda t^{1/\rho};1/\rho)t^{1/\rho-1}E_\rho(\lambda^*(x-t)^{1/\rho};1/\rho)(x-t)^{1/\rho-1}dt= \\ =x^{1/\rho-1}(E_\rho(\lambda x^{1/\rho};1/\rho)-E_\rho(\lambda^*x^{1/\rho};1/\rho))/\lambda-\lambda^*. \tag{1.4}$$

The following classes of functions were introduced in [3]:

1) $C_\alpha^{(\infty)}(\alpha\in[0,1))$ is the class of functions $f(x)$, possessing all derivatives

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$D_\infty^n f(x) \equiv f^{(n)}(x),\ n=0,1,2,\dots$, with $\sup\limits_{0\le x<\infty}\left|(1+x^{m\alpha})f^{(n)}(x)\right|<+\infty,\quad n,m=0,1,\dots;$

2) $C_\alpha^{+(\infty)}(\alpha\in[0,1))$ is the class of continuous functions $f(x)$ on $[0,+\infty)$, possessing all Weil derivatives $D_\infty^{n/\rho}f(x),\ n=0,1,2,\dots$, and satisfying $\sup\limits_{0\le x<+\infty}\left|(1+x^{m\alpha})D_\infty^{n/\rho}f(x)\right|<+\infty,\ n,m=0,1,\dots$ It is proved that $C_\alpha^{(\infty)}, C_\alpha^{+(\infty)}$ coincide [3].

## § 2. Main Results.

***Lemma 2.1.*** Let $\rho\ge 1\,(1-\alpha=1/\rho),\ f(x)\in C_\alpha^{+(\infty)},\ e^{\lambda^\rho x}|f(x)|\in L(0,+\infty).$ Then the following formula holds

$$\int\limits_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int\limits_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)f(t_1)dt_1 = \\ =1/(\lambda-\lambda_1)\int\limits_x^{+\infty} f(t_1)\{e_\rho(t_1-x;\lambda_1)-e_\rho(t_1-x;\lambda_0)\}dt_1, \tag{2.1}$$

where $\lambda_0,\ \lambda_1>0,\ \lambda=\max\{\lambda_0,\lambda_1\},\ e_\rho(x,\lambda)=E_\rho(\lambda x^{1/\rho};1/\rho)x^{1/\rho-1}$.

*Proof.* First note that the inner integral of the left hand side of (2.1) converges absolutely and uniformly on $t_0\in(0,+\infty)$ due to Lemma 2.1 (see [1]). Then the change the order of integration and formula (1.4) yield $\int\limits_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\times$

$$\times\int\limits_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int\limits_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)f(t_1)dt_1=\frac{1}{\lambda_1-\lambda_0}\int\limits_x^{+\infty} f(t_1)\{e_\rho(t_1-x;\lambda_1)-e_\rho(t_1-x;\lambda_0)\}dt_1.$$

***Lemma 2.2.*** Let $\{\lambda_j\}_0^\infty$ $(\lambda_i\ne\lambda_j)$ be an arbitrary sequence of complex numbers.

Then for any $n\ge 1$.

$$\sum_{i=0}^n 1/(\prod_{j=0,\,(i\ne j)}^n(\lambda_j-\lambda_i))=0\,. \tag{2.2}$$

***Lemma 2.3.*** Let $\rho\ge 1\,(1-\alpha=1/\rho)$, $f(x)\in C_\alpha^{+(\infty)}$ and assume that $e^{\lambda_n^\rho x}|f(x)|\in L(0,+\infty)$ for any $n\ge 1$. Then the following formula holds:

$$\int\limits_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int\limits_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)dt_1\dots\int\limits_{t_{n-1}}^{+\infty} e_\rho(t_n-t_{n-1};\lambda_n)f(t_n)dt_n= \\ =\int\limits_x^{+\infty} K_\rho(\tau-x;\{\lambda_i\}_0^n)f(\tau)d\tau,\quad x\in(0,+\infty), \tag{2.3}$$

where

$$K_\rho(x;\{\lambda_i\}_0^n)=\sum_{i=0}^n a_i^{(n)}e_\rho(x;\lambda_i)\,, \tag{2.4}$$

$$a_i^{(n)}=\left\{\prod_{j=0,\,(i\ne j)}^n(\lambda_j-\lambda_i)\right\}^{-1},e_\rho(x;\lambda_i)=E_\rho(\lambda_i x^{\frac{1}{\rho}};1/\rho)x^{\frac{1}{\rho}-1},\ 0<\lambda_i<\lambda_{i+1}<+\infty,\ \ i=0,1,\dots.$$

*Proof.* Let $n=1$. Then, using (2.1), we get

$$\int_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)f(t_1)dt_1=$$
$$=1/(\lambda_0-\lambda_1)\int_x^{+\infty} f(t_1)\{e_\rho(t_1-x;\lambda_0)-e_\rho(t_1-x;\lambda_1)\}dt_1= \tag{2.5}$$
$$=\int_x^{+\infty}\{a_0^{(1)}e_\rho(t_1-x;\lambda_0)+a_1^{(1)}e_\rho(t_1-x;\lambda_1)\}f(t_1)dt_1=\int_x^{+\infty}K_\rho(t_1-x;\{\lambda_i\}_0^1)f(t_1)dt_1,$$

where $K_\rho(t_1-x;\{\lambda_i\}_0^1)=\sum_{i=0}^{1}a_i^{(1)}e_\rho(t_1-x;\lambda_i)$, $a_i^{(1)}=\left\{\prod_{j=0,\,(i\neq j)}^{n}(\lambda_i-\lambda_j)\right\}^{-1}$ $(i=0,1)$.

So, the Lemma is true for $n=1$.

Now we apply the method of mathematical induction. We suppose, the formula is true for $n$, and we prove that it remains true for $n+1$. By our assumption,

$$\int_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)dt_1\int_{t_1}^{+\infty} e_\rho(t_2-t_1;\lambda_2)dt_2\ldots\int_{t_n}^{+\infty} e_\rho(t_{n+1}-t_n;\lambda_{n+1})f(t_{n+1})dt_{n+1}= \tag{2.6}$$
$$=\int_{t_0}^{+\infty}K_\rho(t_1-t_0;\{\lambda_i\}_1^{n+1})f(t_1)dt_1,$$

where $K_\rho(t_1-t_0;\{\lambda_i\}_1^{n+1})=\sum_{i=1}^{n+1}\tilde a_i^{(n+1)}e_\rho(t_1-t_0;\lambda_i)$, $\tilde a_i^{(n+1)}=\left\{\prod_{j=1,\,(i\neq j)}^{n+1}(\lambda_i-\lambda_j)\right\}^{-1}$. (2.7)

Now, using (2.6) and Lemma 2.1, we will have

$$\int_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}K_\rho(t_1-t_0;\{\lambda_i\}_1^{n+1})f(t_1)dt_1=$$
$$=\int_x^{+\infty} e_\rho(t_0-x;\lambda_0)\,dt_0\int_{t_0}^{+\infty}\left\{\sum_{i=1}^{n+1}\tilde a_i^{(n+1)}e_\rho(t_1-t_0;\lambda_i)\right\}f(t_1)dt_1= \tag{2.8}$$
$$=\int_x^{+\infty}f(t_1)\left\{\sum_{i=1}^{n+1}\frac{\tilde a_i^{(n+1)}}{\lambda_0-\lambda_i}e_\rho(t_1-x;\lambda_0)+\sum_{i=1}^{n+1}\frac{\tilde a_i^{(n+1)}}{\lambda_i-\lambda_0}e_\rho(t_1-x;\lambda_i)\right\}dt_1.$$

Next we show that

$$\sum_{i=1}^{n+1}\frac{\tilde a_i^{(n+1)}}{\lambda_0-\lambda_i}=a_0^{(n+1)}=\left\{\prod_{j=1}^{n+1}(\lambda_0-\lambda_j)\right\}^{-1}, \tag{2.9}$$

$$\sum_{i=1}^{n+1}\tilde a_i^{(n+1)}/(\lambda_0-\lambda_i)=\sum_{i=1}^{n+1}\left((\lambda_0-\lambda_i)\prod_{j=1,\,(i\neq j)}^{n+1}(\lambda_i-\lambda_j)\right)^{-1}=-\sum_{i=1}^{n+1}\left(\prod_{j=0,\,(i\neq j)}^{n+1}(\lambda_i-\lambda_j)\right)^{-1} \tag{2.10}$$

(see (2.7)).

According to Lemma 2.2, $\sum_{i=0}^{n+1}1/\prod_{j=0,\,(i\neq j)}^{n+1}(\lambda_i-\lambda_j)=0$, hence,

$$-\sum_{i=1}^{n+1}\left(\prod_{j=0,\,(i\neq j)}^{n+1}(\lambda_i-\lambda_j)\right)^{-1}=\left\{\prod_{j=1}^{n+1}(\lambda_0-\lambda_j)\right\}^{-1}. \tag{2.11}$$

Further we note that

$$\tilde{a}_i^{(n+1)}/(\lambda_i-\lambda_0)=1/\prod_{j=0,\ (i\neq j)}^{n+1}(\lambda_i-\lambda_j)=a_i^{(n+1)}, \qquad (2.12)$$

and using (2.9)–(2.12), we get

$$\int_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)dt_1\ldots\int_{t_n}^{+\infty} e_\rho(t_{n+1}-t_n;\lambda_{n+1})f(t_{n+1})dt_{n+1}=$$

$$=\int_x^{+\infty}\left(\sum_{i=0}^{n+1}a_i^{(n+1)}e_\rho(t_0-x;\lambda_i)\right)f(t_0)dt_0=\int_x^{+\infty}K_\rho(t_0-x;\{\lambda_i\}_0^{n+1})f(t_0)dt_0.$$

The Lemma is proved.

***Theorem.*** Let $f(x)\in C_\alpha^{+(\infty)}$ and $e^{\lambda_n^\rho x}\left|D_\infty^{k/\rho}f(x)\right|\in L(0,+\infty)$, $k=0,1,\ldots,n$ for any $n\geq 1$. Then the following formula holds:

$$f(x)=\sum_{i=0}^{n}L_\infty^{i/\rho}f(0)\varphi_i(x)+(-1)^{n+1}\int_x^{+\infty}K_\rho(\tau-x;\{\lambda_i\}_0^n)L_\infty^{(n+1)/\rho}f(\tau)d\tau, \qquad (2.13)$$

where $K_\rho(x;\{\lambda_i\}_0^n)=\sum_{i=0}^{n}\left\{\prod_{j=0,\ (i\neq j)}^{n}(\lambda_i-\lambda_j)\right\}^{-1}e_\rho(x;\lambda_i),\ \ e_\rho(x;\lambda_i)=E_\rho(\lambda_i x^{\frac{1}{\rho}};\frac{1}{\rho})x^{\frac{1}{\rho-1}}.$

*Proof.* According to Theorem 3.3 (see [1]),

$$f(x)=\sum_{i=0}^{n}L_\infty^{i/\rho}f(0)\varphi_i(x)+R_n(x),\qquad x\in(0,+\infty), \qquad (2.14)$$

where $R_n(x)=(-1)^{n+1}\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)dt_1\ldots\int_{t_{n-1}}^{+\infty}e_\rho(t_n-t_{n-1};\lambda_n)L_\infty^{\frac{n+1}{\rho}}f(t_n)dt_n.$

Since $e^{\lambda_n^\rho x}\left|D_\infty^{k/\rho}f(x)\right|\in L(0,+\infty)$ $(k=0,1,2,\ldots,n)$ for any $n\geq 1$ and $L_\infty^{(n+1)/\rho}f(x)=$ $=\prod_{k=0}^{n}(D_\infty^{k/\rho}+\lambda_k)f(x)$. Then it is easy to see that $e^{\lambda_n^\rho x}\left|L_\infty^{(n+1)/\rho}f(x)\right|\in L(0,+\infty)$.

Now, using Lemma 2.3, we get

$$R_n(x)=(-1)^{n+1}\int_x^{+\infty}K_\rho(\tau-x;\{\lambda_i\}_0^n)L_\infty^{(n+1)/\rho}f(\tau)d\tau. \qquad (2.15)$$

From (2.14) and (2.15) we obtain (2.13).
The Theorem is proved.

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