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Mathematics

SOME ESTIMATES FOR STATIONARY EXTENDED MEAN FIELD GAMES

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Mean field games theory is a recent area of study introduced by Lions and Lasry in a series of seminal papers in 2006. Mean field games model situations of competition between large number of rational agents that play noncooperative dynamic games under certain symmetry assumptions. In this paper we consider quasivariational mean field games system with additional dependence on a velocity field of the players. We obtain certain estimates for the solutions to this system. In a forthcoming paper we intend to obtain an existence result using this kind of estimates and continuation method.

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Introduction. Mean field games theory is a recent area of research started by Caines and his coworkers [1, 2] and independently by Pierre Louis Lions and Jean Michel Lasry [3–6]. It studies differential games involving very large numbers of identical and rational agents. Inspired by ideas in statistical physics, these authors introduced a class of models, in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. There is a fast growing literature on mean field games and its applications, for instance, see [7] and reference therein. Applications of mean field games arise in the study of growth theory in economics [8] or environmental policy [9], in the future they may play an important role in economics and population models. There is also a growing interest in numerical methods forhese problems [9–11]. By D. Gomes and his collaborators the discrete state problem is considered both in discrete time [12] and the continuous time [13]. Several problems have been worked out in details in [14, 15].

Denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$. Let

$$H\colon \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$$

be a function satisfying appropriate continuity, differentiability and growth conditions. Stationary mean field game is a system of PDE's on $\mathbb{T}^d$ of the form

$$\begin{cases} \Delta u(x) + H(x, Du(x), m) = \overline{H}, \\ \Delta m(x) - \operatorname{div}(D_p H(x, Du(x), m)m(x)) = 0. \end{cases}$$

This system of PDE's arise naturally in the study of mean-field games. Stationary mean-field games besides being interesting by themselves, carry also asymptotic properties of time

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dependent mean-field games, as follows from results in [13, 16] concerning the trend to the equilibrium in such problems. A certain type of stationary mean field games can also arise from variational problems: given $H = H(x,p)\colon \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ consider the stochastic Evans–Aronsson problem

$$\int_{\mathbb{T}^d} e^{\varepsilon\Delta\phi + H(x,D\phi(x))}dx \to \inf.$$

The Euler-Lagrange for this functional can be written as

$$\begin{cases} \Delta u(x) + H(x,Du(x)) = \ln m + \overline{H}, \\ \Delta m(x) - \operatorname{div}(D_pH(x,Du(x))m(x)) = 0. \end{cases}$$

In [17] the authors consider so called quasi-variational mean-field games, which are perturbations of mean field games with a variational structure. In this paper we consider an extension of the mean field problem, which allows the cost function of a player to depend also on the actions of the rest of the players, that is on the velocity field of players. Denote by $\chi(\mathbb{T}^d)$ the set of continuous vector fields on $\mathbb{T}^d$. Let

$$H = H(x,p,m,V)\colon \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}.$$

We consider the following equation on the d-dimensional torus $\mathbb{T}^d$:

$$\begin{cases} \Delta u(x) + H(x,Du(x),m,V) = \overline{H}, \\ \Delta m(x) - \operatorname{div}(V(x)m(x)) = 0, \\ V(x) = D_pH(x,Du(x),m,V). \end{cases} \tag{1}$$

The unknowns for this problems are $u\colon \mathbb{T}^d \to \mathbb{R}$, treated as a 1-periodic function on $\mathbb{R}^d$ whenever it is convenient, a probability measure $m \in \mathcal{P}(\mathbb{T}^d)$, an effective Hamiltonian $\overline{H}$ and the effective velocity field $V \in \chi(\mathbb{T}^d)$. We require m to be a probability measure absolutely continuous with respect to the Lebesgue measure. By abuse of notation we use the symbol "m" both for the measure and the density function. We assume that H is C^2 in the variables x and p and satisfies the following growth condition: H is quasi-variational [18], i.e. there exists a concave, increasing function $g\colon (0,\infty) \to \mathbb{R}$ and

$$H_0\colon \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$$

continuous in all its variables, satisfying:

(A1) $$|H(x,p,m,V) - H_0(x,p,m,V) + g(m)| \le C.$$

In the present paper we establish some a-priori estimates for the solutions to this system. This kind of estimates can be later used to prove the existence of the solutions via continuation method.

Assumptions. In this paragraph we introduce and discuss various assumptions that will be required throughout the paper, in addition to (**A1**).

(**A2**) The function $g : (0,\infty) \to \mathbb{R}$ is smooth, strictly increasing and concave with $mg(m)$ convex in m. The following choices of g will be considered separately in various proofs:

a) $g(m) = \ln m$,

b) $g(m) = m^\gamma$ with $0 < \gamma < 1$, and we will refer to them as, respectively, assumptions (A2, a) or (A2, b).

We suppose that for the Hamiltonians $H : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$ and $H_0 : \mathbb{T}^d \times \mathbb{R}^{\times}\mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}$ there exist constants $C, \delta > 0$ such that the following assumptions are satisfied:

(**A3**) For all $(x,p,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \chi(\mathbb{T}^d)$ we get $|p|^2 \leq C + CH_0(x,p,m,V) + +\delta \int_{\mathbb{T}^d} |V|^2 dm$.

For $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$ we define the Lagrangian L associated with H as

$$L(x,p,m,V) = -H(x,p,m,V) + pD_pH(x,p,m,V). \tag{2}$$

Note, that if (u,m,V) solves (1), then

$$L(x,Du,m,V) = -H(x,Du,m,V) + Du \cdot V.$$

With this notation we assume further:

(**A4**) There exists $c > 0$ such that, for all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$L(x,p,m,V) \geq cH_0(x,p,m,V) + g(m) - C - \delta \int_{\mathbb{T}^d} |V|^2 dm.$$

(**A5**) The function $H(x,p,m,V) + g(m(x))$ is twice differentiable in variables x, p, for all absolutely continuous measures $m \in \mathcal{P}(\mathbb{T}^d)$ with density $m(\cdot)$.

(**A6**) For all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|D_pH(x,p,m,V)|^2 \leq C + CH_0(x,p,m,V) + \delta \int_{\mathbb{T}^d} |V|^2 dm.$$

Another hypothesis concerns the convexity of H in p. We suppose:

(**A7**) H is uniformly convex in p: there exists $\gamma > 0$ such that, for all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$D^2_{pp}H(x,p,m,V) \geq \gamma I,$$

where I is the identity matrix in $\mathbb{R}^d$. Set

$$\widehat{H}_{xx} = D_{xx}\left(H(x,p,m,V) + g(m(x))\right);$$
$$\widehat{H}_{xp} = D_x(D_pH(x,p,m,V))$$

and assume that

(**A8**) For all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|\widehat{H}_{xx}(x,p,m,V)| \leq C + CH_0(x,p,m,V) + \delta \int_{\mathbb{T}^d} |V|^2 dm.$$

(**A9**) For all $(x,p,m,V) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \chi(\mathbb{T}^d)$

$$|\widehat{H}_{xp}(x,p,m,V)|^2 \leq C + CH_0(x,p,m,V) + \delta \int_{\mathbb{T}^d} |V|^2 dm.$$

(**A10**) We assume also that $\delta \in [0, \delta_0]$, where δ_0 is given later in Proposition 5.

For example, the following Hamiltonian satisfies all of these conditions:

$$H(x,p,m,V) = \frac{1}{2}|p|^2 - \alpha p \int_{\mathbb{T}^d} V(y)dm(y) - g(m) \quad \text{with} \quad \delta = \frac{\alpha^2}{2}.$$

Estimates. In this paragraph we are going to prove various a-priori estimates for the solutions to the quasivariational stationary extended mean-field game equation (1). Hereafter, by a solution to (1) we mean a classical solution with $m>0$. In this section we use ideas and techniques from [18] with some modifications to allow the dependence of H on V.

Proposition 1. Assume (A1)–(A4) and let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$|\overline{H}| \le C + C\delta \int_{\mathbb{T}^d} |V|^2 dm.$$

Proof. Multiplying the first equation of (1) by m and integrating by parts, we get:

$$\begin{aligned}
\overline{H} &= \int_{\mathbb{T}^d} (\Delta u + H(x,Du,m,V))\, dm = \\
&= \int_{\mathbb{T}^d} (H(x,Du,m,V) - VDu)\, dm = \\
&= -\int_{\mathbb{T}^d} L(x,Du,m,V) dm \le \\
&\le -\int_{\mathbb{T}^d} \left(cH_0(x,Du,m,V) + g(m) dm + \delta \int_{\mathbb{T}^d} |V|^2 \right) dm + C \le \\
&\le C\delta \int_{\mathbb{T}^d} |V|^2 dm + C,
\end{aligned}$$

where we have used (A4), (A3) and (A2) together with the Jensen's inequality and the representation of (2) L.

For the lower estimate, we use the first equation in (1), (A1) and (A2) and obtain

$$\overline{H} \ge \Delta u(x) - g(m) - C - \delta \int_{\mathbb{T}^d} |V|^2 dm.$$

Integrating in x and using again the Jensen's inequality, we get

$$\overline{H} \ge -C - \delta \int_{\mathbb{T}^d} |V|^2 dm. \qquad \square$$

Corollary 1. Assume (A1)–(A4) and let $(u,m,V,\overline{H})$ solve (1). Then there exists $C>0$ such that

$$\int_{\mathbb{T}^d} H_0(x,Du,m,V) dx \le C + C\delta \int_{\mathbb{T}^d} |V|^2 dm.$$

Proof. From the first equation in (1), (A1), Proposition 1 and the Jensen's inequality, we have

$$\int_{\mathbb{T}^d} H_0(x,Du,m,V) dx \le \overline{H} + \int_{\mathbb{T}^d} g(m) dx + C + \delta \int_{\mathbb{T}^d} |V|^2 dm \le C + C\delta \int_{\mathbb{T}^d} |V|^2 dm. \quad \square$$

Corollary 2. Assume (A1)–(A4) and let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$\int_{\mathbb{T}^d} H_0(x,Du,m,V) dm \le C + C\delta \int_{\mathbb{T}^d} |V|^2 dm.$$

Proof. As in the proof of Proposition 1, we have

$$\overline{H} = -\int_{\mathbb{T}^d} L(x,Du,m,V)dm \le C - \int_{\mathbb{T}^d} g(m)dm -$$
$$-C\int_{\mathbb{T}^d} H_0(x,Du,m,V)dm + \delta\int_{\mathbb{T}^d} |V|^2 dm,$$

and using the bounds on $\overline{H}$ from Proposition 1, we finish the Proof. □

Proposition 2. Assume (A1)–(A6). Let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$\|\sqrt{m}\|_{H^1} \le C + C\delta\int_{\mathbb{T}^d} |V|^2 dm.$$

Proof. Multiplying the second equation of (1) by $\ln m$, integrating by parts and using Holder's inequality, we get the estimate

$$\int_{\mathbb{T}^d} \frac{|Dm|^2}{m}dx = \int_{\mathbb{T}^d} DmD_pHdx \le \left(\int_{\mathbb{T}^d} \frac{|Dm|^2}{m}dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} |D_pH|^2 dm\right)^{\frac{1}{2}}.$$

Hence,

$$\int_{\mathbb{T}^d} |D\sqrt{m}|^2 dx = \frac{1}{4}\int_{\mathbb{T}^d} \frac{|Dm|^2}{m}dx \le \frac{1}{4}\int_{\mathbb{T}^d} |D_pH|^2 dm.$$

The result then follows from (A6), Corollary 2 and the equality $\int_{\mathbb{T}^d} mdx = 1$. □

Proposition 3. Assume (A1)–(A6). Let $(u,m,V,\overline{H})$ solve the system (1). Then there exist C, $\delta_0>0$ such that for any $\delta\in[0,\delta_0]$ we have

$$\int_{\mathbb{T}^d} |V|^2 dm \le C.$$

Proof. Using the last equation of (1), (A6) and Corollary 2, we get

$$\int_{\mathbb{T}^d} |V|^2 dm = \int_{\mathbb{T}^d} |D_pH|^2 dm \le C + C\int_{\mathbb{T}^d} H_0 dm + \delta\int_{\mathbb{T}^d} |V|^2 dm \le C + C\delta\int_{\mathbb{T}^d} |V|^2 dm.$$

For small δ this gives the result. □

Combining Proposition 3 with Propositions 1, 2 and Corollaries 1, 2, we get:

Corollary 3. Assume (A1)–(A6) and (A10). Let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$|\overline{H}| \le C, \tag{3}$$
$$\int_{\mathbb{T}^d} H_0(x,Du,m,V)dx \le C,$$
$$\int_{\mathbb{T}^d} H_0(x,Du,m,V)dm \le C,$$
$$\|\sqrt{m}\|_{H^1(\mathbb{T}^d)} \le C.$$

Proposition 4. Assume (A1)–(A10) and let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx \le C, \qquad \int_{\mathbb{T}^d} |D^2u|^2 dm \le C.$$

Proof. Applying the operator Δ on the first equation of (1), we obtain

$$\Delta^2 u + \widehat{H}_{x_i x_i} + 2\widehat{H}_{p_k x_i}(x,Du,m,V)u_{x_k x_i} + Tr(D^2_{pp}H(x,Du,m,V)(D^2u)^2) +$$

$$+D_pH(x,Du,m,V)D\Delta u-\operatorname{div}(g'(m)Dm)=0.$$

Integrating with respect to m, using the integration by parts and the second equation in (1), we get

$$\int_{\mathbb{T}^d}[\widehat{H}_{x_ix_i}+2\widehat{H}_{p_kx_i}(x,Du,m,V)u_{x_kx_i}]dm+\int_{\mathbb{T}^d}g'(m)|Dm|^2dx+$$
$$+\int_{\mathbb{T}^d}Tr(D^2_{pp}H(x,Du,m,V)(D^2u)^2)]dm=0,$$

and by (A7) we deduce

$$\int_{\mathbb{T}^d}g'(m)|Dm|^2dx+\gamma\int_{\mathbb{T}^d}|D^2u|^2dm\leq\int_{\mathbb{T}^d}|\widehat{H}_{x_ix_i}|dm+C|\widehat{H}_{xp}||D^2u|dm.$$

On the other hand, using the Cuachy inequality, we have

$$\int_{\mathbb{T}^d}|\widehat{H}_{xp}||D^2u|dm\leq\frac{\gamma}{2}\int_{\mathbb{T}^d}|D^2u|^2dm+C\int_{\mathbb{T}^d}|\widehat{H}_{xp}|^2dm,$$

implying

$$\int_{\mathbb{T}^d}g'(m)|Dm|^2dx+\frac{\gamma}{2}\int_{\mathbb{T}^d}|D^2u|^2dm\leq\int_{\mathbb{T}^d}[|\widehat{H}_{x_ix_i}|dm+$$
$$+C|\widehat{H}_{xp}|^2]dm\leq C+C\delta\int_{\mathbb{T}^d}|V|^2dm,$$

where in the last inequality we have used (A8) and (A9). Then the Proposition 3 finishes the Proof. □

Corollary 4. Assume (A1)–(A10) and (A2, a), and let $(u,m,V,\overline{H})$ solve the system (1). Then there exists $C>0$ such that

$$\int_{\mathbb{T}^d}m^{\frac{2^*}{2}}dx\leq C \tag{4}$$

and

$$\int_{\mathbb{T}^d}|Du|^4dm\leq C.$$

Proof. From Proposition 4 and $\int mdx=1$ we infer that $m^{\frac{1}{2}}\in H^1(\mathbb{T}^d)$. Then the Sobolev Imbedding Theorem implies (4). Since $\frac{2^*}{2}>1$, from (4) we deduce in particular that

$$\int_{\mathbb{T}^d}g(m)^2dm\leq C. \tag{5}$$

Now, using (A3) and (3), we obtain

$$H_0(x,Du,m,V)\leq C-\Delta u+g(m).$$

Then by (A3), Proposition 4 and (5), we get

$$\int_{\mathbb{T}^d}|Du|^4dm\leq C+C\int_{\mathbb{T}^d}H_0^2dm\leq C+C\int_{\mathbb{T}^d}g(m)^2dm+C\int_{\mathbb{T}^d}|D^2u|^2dm\leq C.$$ □

Corollary 5. Assume (A1)–(A10) and (A2, b). Let $(u,m,V,\overline{H})$ solve (1). Then there exists $C>0$ such that

$$\int_{\mathbb{T}^d}m^{\frac{2^*}{2}(\gamma+1)}dx\leq C. \tag{6}$$

Furthermore, if $2\gamma+1\leq\frac{2^*}{2}(\gamma+1)$, then

$$\int_{\mathbb{T}^d}|Du|^4dm\leq C.$$

Proof. Let $f(x) = m^{\frac{\gamma+1}{2}}(x)$. Since $\gamma < 1$, the mapping $t \mapsto t^{\frac{\gamma+1}{2}}$ is concave, and the Jensen inequality implies

$$0 \leq \int_{\mathbb{T}^d} f dx \leq \left(\int_{\mathbb{T}^d} m dx\right)^{\frac{\gamma+1}{2}} = 1.$$

Note that

$$\int\limits_{\mathbb{T}^d} |Df|^2 dx = C \int\limits_{\mathbb{T}^d} m^{\gamma-1} |Dm|^2 dx = C \int\limits_{\mathbb{T}^d} g'(m) |Dm|^2 dx \leq C.$$

Then, by the Poincaré inequality

$$\int_{\mathbb{T}^d} f^2 dx - \left(\int_{\mathbb{T}^d} f dx\right)^2 = \int_{\mathbb{T}^d} \left(f - \int_{\mathbb{T}^d} f\right)^2 dx \leq C \int_{\mathbb{T}^d} |Df|^2 dx \leq C.$$

Thus, $\|f\|_{H^1} \leq C$. The Sobolev inequality then implies

$$\|f\|_{L^{2^*}} \leq C \|f\|_{H^1} \leq C,$$

which proves (6). In particular, if $2\gamma + 1 \leq \frac{2^*}{2}(\gamma + 1)$, then

$$\int_{\mathbb{T}^d} g(m)^2 m dx = \int_{\mathbb{T}^d} m^{2\gamma+1} dx \leq C + C \int_{\mathbb{T}^d} m^{\frac{2^*}{2}(\gamma+1)} dx = C + C \int_{\mathbb{T}^d} f^{2^*} dx \leq C. \quad (7)$$

Using (A1) and (3), we have

$$H_0(x, Du, m, V) \leq C - \Delta u + g(m).$$

Then by (A3), Proposition 4 and (7)

$$\int_{\mathbb{T}^d} |Du|^4 dm \leq C + C \int_{\mathbb{T}^d} H_0^2 dm \leq C + C \int_{\mathbb{T}^d} g(m)^2 dm + C \int_{\mathbb{T}^d} |D^2 u|^2 dm \leq C.$$

□

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