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Mathematics

ON CRITICAL POINTS OF SOME POLYNOMIALS

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One dynamic property of some polynomials is investigated. The statements about traces of critical points of some polynomials are proved. The equations of curves, on which critical points move, are obtained.

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Introduction. It is well known that all the roots of P', distinct from the multiple roots of the polynomial P itself, are the equilibrium points for the field of forces created by identical particles placed at the roots of P (provided that r particles are located at the root of multiplicity r). This interpretation provides a quick proof of Gauss-Lucas Theorem (see, e.g., [1] or [2]). This kind of problems gave birth to the branch of mathematics, which, after the book of Morris Marden [3], was called Geometry of Polynomials. The polynomial conjectures of Sendov and Smale are two challenging problems of this branch [4–6].

Marden's Theorem [3, 7] gives a geometric relationship between the zeros of a third-degree polynomial with complex coefficients and the zeros of its derivative. A more general version of this Theorem is due to Linfield [8].

This article focuses on the dynamic behavior of critical points of cubic polynomial in the case when one of its roots moves along a given trajectory and two others are fixed. We describe the geometric place of critical points of this polynomials and give the relationship between zeros and critical points of such polynomials. The case of multiple roots of the given polynomial is considered as well. The proofs of the statements can easily be done, so we omit them.

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The Case of Simple Roots.

Proposition 1. Let $z_1 = 0$, $z_2 = 1$ and $z_3 = e^{i\varphi}$, where the angle φ runs from 0 to 2π, be the roots of a complex polynomial of the third degree $P(z)$. Then the geometrical place of critical points of this polynomial consists of two circles given by the equations

$$|z| = \frac{1}{\sqrt{3}} \tag{1}$$

and

$$\left|z - \frac{2}{3}\right| = \frac{1}{3}. \tag{2}$$

Proposition 2. The geometric place of critical points of the polynomial $P(z)$ with roots $(0, 1,\ e^{i\varphi})$ is the intersection of circle (2) and the straight line $y=\mathrm{tg}\frac{\varphi}{2}x$ when $|\varphi| \le \frac{\pi}{3}$, and the intersection of circle (1) and the straight line $y=\mathrm{ctg}\frac{\varphi}{2}\left(\frac{2}{3} - x\right)$ when $\frac{\pi}{3} < \varphi < \frac{5\pi}{3}$.

Proposition 3. Suppose that $z_1 = 0$ and $z_2 = 1$ are two fixed roots of a cubic polynomial $P(z)$, and the third root z_3 moves along the perpendicular bisector of the segment [0, 1]. Then:

a) the geometric locus of critical points of $P(z)$, for z_3 such that $|\mathrm{Im}\, z_3| \le \frac{\sqrt{3}}{2}$, is the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{\sqrt{3}}{6}\right)^2$, which intersection with a straight line $y = \frac{\mathrm{Im}\, z_3}{3}$ determines the location of critical points of $P(z)$;

b) for z_3 satisfying $|\mathrm{Im}\, z_3| > \frac{\sqrt{3}}{2}$, the geometric locus of critical points of $P(z)$ is the perpendicular bisector of the segment [0, 1]. The intersection this bisector with a circle, the center of which divides the segment $|z_1 - z_3|$ in the ratio of $1 : 3$ measured from the point of z_1, and with a radius equal to $\frac{|z_1 - z_3|}{3}$, determines the location of critical points of $P(z)$.

The Case of Multiple Roots.

Proposition 4. The geometric place of critical points of polynomial

$$P(z) = z^{\alpha}(z-1)^{\beta}(z - e^{i\varphi})^{\beta},$$

where the angle φ runs from 0 to 2π, consists of two circles given by the equations

$$|z| = \sqrt{\frac{\alpha}{\alpha + 2\beta}} \tag{3}$$

and

$$\left|z - \frac{\alpha+\beta}{\alpha+2\beta}\right| = \frac{\beta}{\alpha+2\beta}. \tag{4}$$

Proposition 5. The geometric place of critical points of the polynomial $P(z)=z^{\alpha}(z-1)^{\beta}(z-e^{i\varphi})^{\beta}$ is the intersection of circle (3) and the straight line $y=\mathrm{tg}\frac{\varphi}{2}x$ when $|\varphi|\leq\frac{\alpha^2+2\alpha\beta-\beta^2}{(\alpha+\beta)^2}$, and the itersection of circle (4) and the straight line $y=\mathrm{ctg}\frac{\varphi}{2}\left(\frac{\alpha+\beta}{\alpha+2\beta}-x\right)$ when

$$\arccos\frac{\alpha^2+2\alpha\beta-\beta^2}{(\alpha+\beta)^2}<\varphi<2\pi-\arccos\frac{\alpha^2+2\alpha\beta-\beta^2}{(\alpha+\beta)^2}.$$

Proposition 6. Suppose that $z_1=0$ and $z_2=1$ are two fixed roots of polynomial $P(z)$ of multiplicity m, and the third root z_3, of multiplicity k moves along the perpendicular bisector of the segment [0, 1]. Then for the critical points of a polynomial $P(z)$, different from the points z_1, z_2 and z_3, we have :

a) the geometric locus of such critical points of $P(z)$, for z_3 satisfying $|\mathrm{Im}z_3|\leq\frac{\sqrt{k^2+mk}}{2m}$, is the circle $\left(x-\frac{1}{2}\right)^2+y^2=\left(\frac{\sqrt{k^2+2mk}}{2(2m+k)}\right)^2$, and the intersection of this circle with the straight line

$$y=\frac{m\mathrm{Im}\,z_3}{2m+k}$$

determines the location of prime critical points of $P(z)$;

b) for z_3 satisfying $|\mathrm{Im}z_3|>\frac{\sqrt{k^2+2mk}}{2m}$, the geometric locus of critical points of $P(z)$ is the perpendicular bisector of the segment [0, 1]. The intersection of this bisector with a circle, the center of which divides the segment $|z_1-z_3|$ in the ratio of $m:2m+k$ measured from the point of z_1, and with a radius equal to $\frac{m|z_1-z_3|}{2m+k}$, determines the location of such critical points of $P(z)$.

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