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Mathematics

SOLUTION OF ONE VOLTERRA TYPE NONLINEAR INTEGRAL EQUATION ON POSITIVE SEMI-AXIS

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The work is devoted to the investigation of one class of Volterra type nonlinear integral equations on positive half-line. The specified class of equations except for self-interest in mathematics has also important applications in physical kinetics. The combination of special factorization methods with methods of construction of invariant cone segments allows us to construct a non-negative solution of initial equation, and investigate integral asymptotics of that solution at infinity.

MSC2010: 45G05.

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Introduction and Formulation of the Main Result. Nonlinear integral equations of the form

$$\varphi(x) = G\Big(x, g(x) + \int_0^x v(x,t)\varphi(t)dt\Big), \quad x \geq 0, \tag{1}$$

with respect to the unknown measurable function $\varphi(x)$ arise in several problems of mathematical physics. In particular, by means of equation (1) are described model problems of kinetic theory of gases (see [1]). In the particular case when kernel $v(x,t)$ depends on the difference of arguments: $v(x,t) = v(x-t)$, an equation of this type are also arise in the theory of Markov processes, and in theory of renewal equations. (see [2]). In the latter case there are numerous works devoted to the study of such equations (see [3, 4] and citation therein). These papers are mainly devoted to the problems of solvability of the equation (1) in $L_p(R^+), p > 1$ spaces in case, when the function $G(x,z)$ satisfies Hölder–Lipschitz condition by second argument with certain power, monotonicity condition by z and some technical conditions are assumed. It was also assumed that the following conditions are fulfilled

$$0 \leq g \in L_p(R^+), \;\; p > 1, \;\; v(x) \geq 0, \;\; x \geq 0, \;\; \int_0^\infty v(\tau)d\tau \leq 1.$$

In the present work under certain conditions on the functions G, V and g, the solvability of equation (1) in $L_1^{loc}(R^+), \;\; R^+ \equiv [0, +\infty)$ is studied.

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Here we assume that the following conditions are fulfilled.

$A)$ $0 \le g \in L_1(R^+)$, $m_1(g) \equiv \int_0^\infty xg(x)dx < +\infty$, $g(x) \not\equiv 0$, $x \in R^+$;

$B)$ the kernel $v(x,t)$ has the form

$$v(x,t) = \int_a^b \alpha(t,s)e^{-\alpha(t,s)(x-t)}d\sigma(s) \cdot \Theta(x-t), \tag{2}$$

where $\alpha(t,s)$ is defined on $R^+ \times [a,b]$, measurable function $(0 \le a \le b \le +\infty)$ satisfies the following conditions:

$B_1)$ there exists a number $\varepsilon_0 > 0$, such that

$$\alpha(t,s) \ge \varepsilon_0 > 0, \ \ t \in R^+, \ \ s \in [a,b); \tag{3}$$

$B_2)$ there exists number $\beta \in (0, \varepsilon_0)$ such that

$$\delta \equiv \sup_{t \in R^+} \int_a^b \Big(1 - \frac{\beta}{\alpha(t,s)}\Big) d\sigma(s) < 1. \tag{4}$$

Here $\sigma(s)$ is a non decreasing function on $[a,b)$ with

$$\int_a^b d\sigma(s) = 1, \tag{5}$$

and Θ is the well known Heaviside function.

$C)$ $G(x,z)$ is a measurable function, defined on $R^+ \times R^+$ satisfying the following conditions:

$C_1)$ for each fixed $x \in R^+$, G is increasing with respect to z, when $z \ge g(x)$;

$C_2)$ $G(x,z)$ satisfies Caratheodory's condition with respect to the argument z on the set $R^+ \times R^+$, i.e. for each fixed $z \in R^+$, the function $G(x,z)$ is measurable in x, and for almost all $x \in R^+$, $G(x,z)$ is continuous in z on R^+;

$C_3)$ for each fixed $x \in R^+$,

$$0 \le G(x,z) \le z, \ \ z \ge g(x). \tag{6}$$

The main result of the present work is the following:

Theorem. Let conditions $A)$, $B)$, $B_1)$, $B_2)$, $C_1) - C_3)$ are fulfilled. Then the equation (1) has a nontrivial, non-negative solution in $L_1^{loc}(R^+)$ with integral asymptotics $\int_0^x \varphi(t)dt = O(x)$, as $x \to +\infty$.

Proof of the Theorem.

We consider the following auxiliary linear Volterra equation

$$\mathcal{P}(x) = g(x) + \int_0^x v(x,t)\mathcal{P}(t)dt, \ \ x \ge 0 \tag{7}$$

with respect to the unknown function $\mathcal{P}(x)$, where the kernel $v(x,t)$ is given by (2).

Note that conditions (2), (3) and (5) imply that

$$\sup_{t \ge 0} \int_0^\infty v(x,t)dx = 1. \tag{8}$$

The equation (8) in a certain way makes difficult to deduce the existence of a non-negative solution for equation (7).

We denote by $\mathfrak{M}$ the class of the following Volterra type integral operators: $\hat{V}_0 \in \mathfrak{M}$, if

$$(\hat{V}_0 f)(x) = \int_0^x v_0(x,t) f(t) dt, \;\; x \in R^+, \;\; f \in L_1^{loc}(R^+), \tag{9}$$

and

$$v_0(x,t) \geq 0, \;\; (x,t) \in R^+ \times R^+, \;\; \mu(v_0) \equiv \sup_{t \in R^+} \int_0^\infty v_0(x,t) dx < +\infty. \tag{10}$$

The equation (7) can be rewritten in the operator form as

$$(I - \hat{V})\mathcal{P} = g, \tag{11}$$

where I is the identity operator, and $\hat{V}$ is the Volterra integral operator with kernel (2). From (8) and (2) it follows that $\hat{V} \in \mathfrak{M}$. We consider the following factorization problem: for a given operator $\hat{V} \in \mathfrak{M}$ (with kernel (2)) find an operator $\hat{W} \in \mathfrak{M}$ such that the following factorization is true:

$$I - \hat{V} = (I - \hat{U})(I - \hat{W}), \tag{12}$$

where $\hat{U} \in \mathfrak{M}$ has the following structure:

$$(\hat{U} f)(x) = \beta \int_0^x e^{-\beta(x-t)} f(t) dt, \;\; f \in L_1^{loc}(R^+). \tag{13}$$

The factorization (12) should be understood as an equality of operators, acting in $L_1^{loc}(R^+)$.

From (12) is follows that

$$\hat{W} = \hat{V} - \hat{U} + \hat{U}\hat{W}. \tag{14}$$

In terms of the kernels, (14) is equivalent to

$$w(x,\tau) = v(x,\tau) - u(x,\tau) + \int_\tau^x u(x,t) w(t,\tau) dt, \;\; (x,\tau) \in R^+ \times R^+, \tag{15}$$

where $u(x,\tau) = \beta e^{-\beta(x-\tau)} \Theta(x-\tau)$, and $w(x,\tau)$ is the kernel of the integral operator $\hat{W} \in \mathfrak{M}$.

It is easy to check that the function

$$w(x,\tau) = \int_a^b (\alpha(\tau,s) - \beta) e^{-\alpha(\tau,s)(x-\tau)} d\sigma(s) \cdot \Theta(x-\tau) \tag{16}$$

satisfies (15) and

$$w(x,\tau) \geq 0, \qquad \mu(w) \leq \delta < 1, \tag{17}$$

where δ is determined by means of the formula (4). Therefore, $\hat{W} \in \mathfrak{M}$. Below we verify that the equation (15) has a unique solution in the following class of functions:

$$\Omega = \{ w(x,\tau) \geq 0, \;\; (x,\tau) \in R^+ \times R^+, \;\; \mu(w) < +\infty \}, \tag{18}$$

where μ is defined in (10).

Let $w_1, w_2 \in \Omega$ be the two different solutions of equation (15). Due to linearity of (15), their difference $\Delta w = w_1 - w_2$ satisfies corresponding homogeneous equation

$$\Delta w(x,\tau) = \int_\tau^x u(x,t) \Delta w(t,\tau) dt. \tag{19}$$

Since $\mu(|\Delta w|) < +\infty$, then for each $\gamma > 0$

$$\int_0^\infty e^{-\gamma x} |\Delta w(x,\tau)| dx \le \mu(|\Delta w|) < +\infty.$$

By multiplying both sides of (19) by $e^{-\gamma x}, \gamma > 0, x \in R^+$ and integrating both sides of the obtained equality in x from τ to $+\infty$, we will obtain from (19)

$$\int_\tau^\infty e^{-\gamma x} \Big|\Delta w(x,\tau)\Big| dx \le \int_\tau^\infty e^{-\gamma x} \int_\tau^x u(x,t) \Big|\Delta w(t,\tau)\Big| dt dx.$$

By changing the order of integration and due to Fubini's theorem, we get

$$\int_\tau^\infty e^{-\gamma x} \Big|\Delta w(x,\tau)\Big| dx \le \int_\tau^\infty \Big|\Delta w(t,\tau)\Big| \int_t^\infty e^{-\gamma x} u(x,t) dx dt,$$

or (taking into account the representation of $u(x,t)$)

$$\int_\tau^\infty e^{-\gamma x} \Big|\Delta w(x,\tau)\Big| dx \le \frac{\beta}{\beta+\gamma} \int_\tau^\infty e^{-\gamma x} \Big|\Delta w(t,\tau)\Big| dt,$$

which implies that $\Delta w(x,\tau) = 0$ almost everywhere on set $R^+ \times R^+$. Thus, the uniqueness of solution of equation (15) in Ω is proved. Therefore, the factorization (12) is unique in the class of integral operators $\mathfrak{M}$, and the kernel of operator $\hat{W} \in \mathfrak{M}$ is given by (16).

Now, by means of factorization (12), we will prove that equation (7) has non-negative solution in $L_1^{loc}(R^+)$, and $\int_0^x \mathcal{P}(t)dt = O(x)$ as $x \to +\infty$.

Taking into account (12), the equation (11) can be rewritten as

$$(I - \hat{U})(I - \hat{W})\mathcal{P} = g, \tag{20}$$

The solution of equation (20) is reduced to successive solution of the following two coupled equations:

$$(I - \hat{U})F = g, \tag{21}$$

$$(I - \hat{W})\mathcal{P} = F. \tag{22}$$

Equation (21) can be represented as

$$F(x) = g(x) + \beta \int_0^x e^{-\beta(x-t)} F(t) dt, \ \ x \ge 0. \tag{23}$$

It is obvious that the solution of (23) has the form

$$F(x) = g(x) + \beta \int_0^x g(t) dt, \ \ x \ge 0. \tag{24}$$

Since $g \in L_1(R^+)$, then from (24) it follows that

$$F \in L_1^{loc}(R^+). \tag{25}$$

Next, concerning the solution of equation (22)

$$\mathcal{P}(x) = F(x) + \int_0^x w(x,t)\mathcal{P}(t) dt, \ x \ge 0, \tag{26}$$

we consider the following successive approximation for (26)

$$\mathcal{P}_{n+1}(x) = F(x) + \int_0^x w(x,t)\mathcal{P}_n(t) dt, \mathcal{P}_0 \equiv 0, \ \ n = 0,1,2,..., \ \ x \ge 0. \tag{27}$$

By means of induction it is easy to verify that:

1)$\mathcal{P}_n(x) \uparrow$ in n;

2)$\mathcal{P}_n \in L_1^{loc}(R^+)$, $n = 1, 2, 3, ...$;

3) for each $r > 0$ the following inequality $\int_0^r \mathcal{P}_n(t)dt \leq (1-\delta)^{-1} \int_0^r F(t)dt, \qquad n = 0, 1, 2, ...$ is fulfilled.

Therefore, from B.Levi's theorem (see [5]), due to arbitrariness of r it follows the existence of a function $\mathcal{P} \in L_1^{loc}(R^+)$ such that $\mathcal{P}_n \uparrow \mathcal{P}$, when $n \to \infty$, in $L_1^{loc}(R^+)$ and $\mathcal{P}$ satisfies the equation (26) and the following inequality

$$\int_0^r \mathcal{P}(t)dt \leq (1-\delta)^{-1} \int_0^r F(t)dt. \tag{28}$$

Taking into consideration (24), from (26) and (28), we get

$$\frac{1}{r}\int_0^r F(t)dt \leq \frac{1}{r}\int_0^r \mathcal{P}(t)dt \leq \frac{(1-\delta)^{-1}}{r}\int_0^r F(t)dt, \tag{29}$$

where

$$\frac{1}{r}\int_0^r F(t)dt = \frac{1}{r}\left(\int_0^r g(t)dt + \beta\int_0^r\int_0^t g(u)dudt\right) =$$

$$= \frac{1}{r}\left(\int_0^r g(t)dt + \beta\int_0^r g(u)(r-u)du\right) \to \beta\int_0^\infty g(u)du, \tag{30}$$

when $r \to +\infty$, since $g \in L_1(R^+)$, $m_1(g) < +\infty$.

Therefore, taking into account (30), from (29) we will obtain the following asymptotic equality

$$\int_0^x \mathcal{P}(t)dt = O(x), \ \ x \to +\infty. \tag{31}$$

Now, we turn to the solution of the initial equation (1). For this purpose we consider the following iteration:

$$\varphi_{n+1}(x) = G\Big(x, g(x) + \int_0^x v(x,t)\varphi_n(t)dt\Big), \ x \geq 0, \ \varphi_0(x) \equiv 0, \ \ n = 0, 1, 2, ... \tag{32}$$

Using conditions C_1) and C_3) and applying induction method it is easy to check that:

1) $\varphi_n(x) \uparrow$ in n;

2) $\varphi_n(x) \leq \mathcal{P}(x)$, $n = 0, 1, 2, ...,$ $x \geq 0$,

where $\mathcal{P}(x)$ is the solution of equation (7). Therefore, the sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ has a pointwise limit, as $n \to \infty$: $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$, where the function $\varphi(x)$ satisfies the following inequality

$$G(x, g(x)) \leq \varphi(x) \leq \mathcal{P}(x), \ \ x \geq 0. \tag{33}$$

From condition C_2) by the B. Levi's theorem, we conclude that $\varphi(x)$ satisfies equation (1).

Since $0 \leq \varphi(x) \leq \mathcal{P}(x)$, then from (31) it follows that

$$\int_0^x \varphi(t)dt = O(x).$$

□

At the end of the paper let give the following two remarks:

Remark 1. If function $\alpha(t,s)$ satisfies condition (3) and

$$\inf_{t\in R^+}\int_a^b \frac{1}{\alpha(t,s)}d\sigma(s)>0, \tag{34}$$

the condition (4) is automatically satisfied.

As to the condition (34), it will be satisfied for example, when α is also bounded above.

Remark 2. To illustrate the result, we give two examples of function $G(x,z)$, satisfying $C_1)-C_3)$:

1. $G(x,z)=z^q(g(x))^{1-q},\;\; q\in(0,1),\; z\geq 0,\; g(x)\geq 0,$

2. $G(x,z)=g(x)\ln\left(\frac{z}{g(x)}+1\right).$

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