PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2013, № 3, p. 18–22

Mathematics

ALGEBRA OF HYPER-ANALYTIC FUNCTIONS AS A β-UNIFORM ALGEBRA

T. M. KHUDOYAN *

Chair of Differential Equations YSU, Armenia

The present paper is devoted to the β-uniform algebra of bounded generalized analytic functions on the "generalized disk". The issues related to the well-known corona problem for this topological algebra are investigated.

MSC2010: 46H05; 46H20.

***Keywords*:** β-uniform, algebra, corona, "generalized disk", topology.

Let Γ be an additive semigroup of real numbers group $\mathbb{R}$ equipped with discrete topology, while $G = \hat{\Gamma}$ [1,2]. Then G is a compact Abelian group. Denote by $\mathbb{C}_\Gamma$ the locally compact space obtained from the Cartesian product $G\times[0,\infty)$ by means of identifying a layer $G\times\{0\}$ with a point (i.e. $\mathbb{C}_\Gamma = G\times[0,\infty)\big/ G\times\{0\}$ with the point $\{*\} = G\times\{0\}\big/ G\times\{0\}$). Observe that when $\Gamma=\mathbb{Z}$, then $G=\mathbb{T}=\{z\in\mathbb{C} : |z|=1\}$, while $\mathbb{C}_\Gamma=\mathbb{C}$ is the complex plane.

The set $\Delta_\Gamma = G\times[0,1)\big/ G\times\{0\}$ is called the "generalized disk". Note that when $\Gamma=\mathbb{Z}$, we obtain $\Delta_\Gamma=\Delta=\{z\in\mathbb{C} : |z|<1\}$.

Recall that for the compact Abelian group G, by the Duality Theorem of L.S. Pontryagin, Γ is the character group for the group G. Denote by $\Gamma_+ = \{a \in \Gamma : a \geqslant 0\}$ the semigroup in the character group Γ. For every character $a\in\Gamma_+$ we consider continuous functions $\chi_a(\alpha r) = \alpha(a) r^a$ ($0^a = 0$ on $\mathbb{C}_\Gamma$, while $\alpha(a)$ is the value of $\alpha\in G$ at $a\in\Gamma$). The obtained family of functions $\langle\chi_a\rangle_{\Gamma_+}$ separates the points of space $\mathbb{C}_\Gamma$.

Let $\mathcal{D}\subset\mathbb{C}_\Gamma$ be an open set, and let f be a continuous function on $\mathcal{D}$. Then one says that f is a generalized analytic function on $\mathcal{D}$, if it is locally, in some neighborhood of each point of $\mathcal{D}$, can be approximated by linear combinations of functions from $\{\chi_a\}_{\Gamma_+}$. Recall that a finite linear combination of functions from $\{\chi_a\}_{\Gamma_+}$ is a polynomial, while two polynomials ratio is a rational function.

Observe that, depending on choice of Γ, one obtains different spaces $\mathbb{C}_\Gamma$.

For a compact $K\subset\mathbb{C}_\Gamma$ hereafter by $P(K)$ we will denote the uniform algebra generated by polynomials on K and by $R(K)$ a uniform algebra generated by rational functions $P_1\big/P_2$, where zeros of polynomial P_2 lie out of K.

* E-mail: tigrankhudoyan@mail.ru

Recall that for a compact $K \subset \mathbb{C}_\Gamma$ the sets

$$K^0 = \{s \in \mathbb{C}_\Gamma : |p(s)| \leqslant \sup_K |p|, p \in \mathcal{P}\}$$

and

$$K_0 = \left\{ s \in \mathbb{C}_\Gamma : \left| p_1(s) \big/ p_2(s) \right| \leqslant \sup_K |p_1/p_2|, p_1, p_2 \in \mathcal{P} \right\}$$

are called polynomially convex hull and rationally convex hull respectively for K, and $K \subset K_0 \subset K^0$. Here $\mathcal{P}$ stands for the family of all polynomials.

It is well known [1,2] that for given algebras maximal ideal spaces $M_{P(K)} = K^0$, while $M_{R(K)} = K_0$ and, if the group Γ is isomorphic to some subgroup of rational number group Q, then we have $K = K_0$ for every compact $K \subset \mathbb{C}_\Gamma$.

We denote by $\mathcal{A}(G)$ the uniform algebra on the group G, which is generated by characters $\{\chi_a\}_{\Gamma_+}$. Then the local structure of generalized analytic functions [2] is as follows: for each point of the space $\mathbb{C}_\Gamma \setminus \{*\}$ there exists its closed neighborhood F homeomorphic to the Cartesian product of $G_a \times W$ such that the uniform algebra $P(F) \cong \mathcal{A}(G_a \times W)$ (i.e. isomorphically isometric), where the algebra $\mathcal{A}(G_a \times W)$ is the uniform algebra on $G_a \times W$, generated by functions of the form fg, where $f \in P(W)$, $g \in C(G_a)$. Here $G_a = \{\alpha \in G : \alpha(a) = 1\}$ and $W \subset \mathbb{C}$ is a compact.

For the sequel recall that $H^\infty(\Delta)$ is the classic uniform algebra of bounded analytic functions on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with respect to sup-norm (i.e. $\|f\|_\infty = \sup_\Delta |f|$), while $H(\Delta)$ is the commutative algebra of all analytic functions on Δ, which is Frechet algebra.

In what follows, we will assume that group $\Gamma = Q_d$, where Q_d is the group of all real rational numbers equipped with discrete topology. Then $G = \hat{Q}_d$ is a connected compact group of characters on the group $\Gamma = Q_d$. Consider the "generalized disk" $\Delta_\Gamma = G \times [0,1) \big/ G \times \{0\}$ with point $\{*\} = G \times \{0\} \big/ G \times \{0\}$. For each $q \in \Gamma_+ = Q \cap [0, \infty)$, the character $\chi_q(g) = g(q)$ on G can be extended to $\bar{\Delta}_\Gamma$ by the formula:

$$\tilde{\chi}_q(\lambda, g) = \begin{cases} \lambda^q \chi_q(g), & \text{when} \quad q \neq 0 \text{ and } \lambda \neq 0, \\ 0, & \text{when} \quad x = *,\ q \neq 0, \\ 1, & \text{when} \quad q = 0. \end{cases}$$

In the case when $G = \hat{Q}_d$, the following conditions are equivalent: the continuous function f on $\bar{\Delta}_\Gamma$ is a generalized analytic function, if and only if f is uniformly approximable on $\bar{\Delta}_\Gamma$ by hyper-polynomials, i.e. by finite linear combinations of functions $\left\{\tilde{\chi}_q\right\}$, $q \in \Gamma_+$ [1,3].

In connection with this, continuous function f on Δ_Γ is called hyper-analytic, if it can be uniformly approximated on Δ_Γ by functions of the form $h \circ \tilde{\chi}_{1/m}$, where $m \in \mathbb{Z}_+$ and $h \in H(\Delta)$.

We denote by $H^\infty(\Delta_\Gamma)$ [3] the uniform algebra of bounded hyper-analytic functions with the sup-norm (i.e. $\|f\|_\infty = \sup_{\Delta_\Gamma} |f(\lambda, g)| = \sup_{\substack{|\lambda|<1 \\ g \in G}} |f(\lambda, g)|$). By $M_{H^\infty(\Delta_\Gamma)}$ we denote the maximal ideal space of the algebra $H^\infty(\Delta_\Gamma)$. It is obvious that each point $(\lambda, g) \in \Delta_\Gamma$ generates a linear multiplicative functional

$\varphi_{(\lambda,g)} \in M_{H^\infty(\Delta_\Gamma)}$ by formula $\varphi_{(\lambda,g)}(f) = f(\lambda,g)$, where $(0,g) = \{*\}$ for each $g \in G$. As in the classical case, the Gelfand transform $f \to \hat{f}$ is an isometric isomorphism between $H^\infty(\Delta_\Gamma)$ and $\hat{H}^\infty(\Delta_\Gamma)$, since if $\hat{f} \equiv 0$, then $\hat{f}(\varphi_{(\lambda,g)}) = f(\lambda,g) = 0$ for every λ with $|\lambda| < 1$ and $g \in G$, i.e. $f \equiv 0$ and $\|\hat{f}\|_\infty = \|f\|_\infty$. Therefore, the "generalized disk" Δ_Γ is embedded into $M_{H^\infty_{(\Delta_\Gamma)}}$, and Δ_Γ is dense in $M_{H^\infty_{(\Delta_\Gamma)}}$. Thus, algebra $H^\infty(\Delta_\Gamma)$ does not possess a corona [3, 4].

Recall some basic notions on β-uniform algebras, which we will use below.

Let $\Omega = \bigcup\limits_{n=1}^{\infty} K_n$ be a local compact space, which admits a compact exhaustion, and where $K_n \subset K_{n+1}$ and every K_n is a compact.

Recall that a closed subalgebra $\mathcal{A}$ of the algebra $C_\beta(\Omega)$ is called β-uniform, if it contains constants and separates points of Ω, i.e. for any $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ there exists a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$ [5–7].

The topology of the sup-norm is stronger than β-uniform topology, hence, β-uniform algebra is a closed subalgebra of the Banach algebra $C_b(\Omega)$ in the sup-norm. The algebra $\mathcal{A}$ equipped with β-uniform topology hereafter we will denote by $\mathcal{A}_\beta$ (or $\mathcal{A}_\beta(\Omega)$). We denote the algebra $\mathcal{A}$ equipped with the topology of sup-norm by $\mathcal{A}_\infty$ (or $\mathcal{A}_\infty(\Omega)$), while its maximal ideal space will be denoted by $M_{\mathcal{A}_\infty}$. Observe that $M_{C_b(\Omega)}$ is the Stone-Cech compactum for Ω.

According to the I.M. Gelfand theory [8, 9], each multiplicative functional on the algebra $\mathcal{A}_\infty(\Omega)$ is continuous in sup-norm, while $M_{\mathcal{A}_\infty}$ is a compact in $*$-weak topology of the subset $S(\mathcal{A}^*) = \{\varphi \in \mathcal{A}^*_\infty : \|\varphi\| = 1\}$. We denote by $M_{\mathcal{A}_\beta}$ the set of all β-continuous linear multiplicative functionals of the β-uniform algebra $\mathcal{A}_\beta(\Omega)$. It is clear, that $M_{\mathcal{A}_\beta} \subset M_{mathcal A_\infty}$. We denote the Shilov boundary of the algebra $\mathcal{A}_\infty$ by $\partial\mathcal{A}_\infty$. The set $\partial\mathcal{A}_\infty \cap \Omega$ will be called the Shilov β-boundary for the algebra $\mathcal{A}_\beta$ and denoted by $\partial\mathcal{A}_\beta$.

As it was shown in [10], if $\partial\mathcal{A}_\infty(\Omega) = \partial\mathcal{A}_\beta(\Omega)$, then $M_{\mathcal{A}_\beta} = M_{\mathcal{A}_\infty}$, i.e. in this case all multiplicative functionals on the algebra $\mathcal{A}_\beta(\Omega)$ are continuous.

Let $C_\beta(\Delta_\Gamma)$ be the β-uniform algebra of all bounded β-topology complex-valued functions on Δ_Γ, then the following statement is true:

Theorem 1. $H^\infty_\beta(\Delta_\Gamma)$ is a β-uniform subalgebra of $C_\beta(\Delta_\Gamma)$.

Proof. Let $\{f_i\}_{i\in I} \subset H^\infty(\Delta_\Gamma)$ be a Cauchy β-net. Then it converges in the β-topology to some continuous and bounded function f on Δ_Γ. For an arbitrary point $(\lambda_0, g_0) \in \Delta_\Gamma$, let U be a neighborhood of (λ_0, g_0) such that $\bar{U} \subset \Delta_\Gamma$, where $\bar{U}$ is the closure of U in Δ_Γ. By Urysohn's Lemma, there exists a function $g \in C_0(\Delta_\Gamma)$, which is equal to the unity on U. Definition of β-topology implies that the net $\{gf_i\}_{i\in I}$ converges uniformly to some function f on Δ_Γ. Hence, we obtain that the net of hyper-analytic functions $\{f_i\}_{i\in I}$ converges uniformly to f on U. Hence, f is a hyper-analytic function on U, and the function $f \in H^\infty(\Delta_\Gamma)$ due to arbitrariness of the point $(\lambda_0, g_0) \in \Delta_\Gamma$. □

Theorem 2. $M_{H^\infty_\beta(\Delta_\Gamma)} = \Delta_\Gamma$.

Proof. Assume that $\varphi \in M_{H^\infty_\beta(\Delta_\Gamma)}$, then it will be a continuous functional on the Banach algebra $H^\infty(\Delta_\Gamma)$. $H^\infty(\Delta_\Gamma)$ is a logmodular algebra [4, 5], hence, there exists a finite, regular measure μ_φ for $\varphi \in M_{H^\infty(\Delta_\Gamma)}$, which is a unique representation measure for φ, i.e.

$$\varphi(f) = \int\limits_{M_{H^\infty(\Delta_\Gamma)}} \hat{f}\, d\mu_\varphi\,,$$

where $\hat{f}$ is the Gelfand transform of the functions f. We represent the measure μ_φ as $\mu_\varphi = \mu_{\Delta_\Gamma} + \mu_{F_\infty}$, where $M_{H^\infty(\Delta_\Gamma)} = \Delta_\Gamma \cup F_\infty$, while μ_Δ and μ_{F_∞} are the restrictions of the measure μ_φ on Δ_Γ and F_∞ respectively. We verify that $\mu_{F_\infty} = 0$. Since $C_0(\Delta_\Gamma)$ possesses a bounded approximative unity $\{e_i\}_{i\in I}$, then the net $\{f_i\}_{i\in I}$, where $f_i = 1 - e_i$, converges by β-topology in $H^\infty_\beta(\Delta_\Gamma)$ to zero function on Δ_Γ. Therefore, the functional net $(f_i \circ \varphi)_{i\in I}$, where $(f_i \circ \varphi)(f) = \varphi(f_i f)$, converges to zero functional. Thus,

$$0 = \lim_I (f_i \circ \varphi)(f) = \lim_I \left(\int\limits_{\Delta_\Gamma} \widehat{f_i f}\, d\mu_\varphi + \int\limits_{F_\infty} \widehat{f_i f}\, d\mu_\varphi \right) = \int\limits_{F_\infty} \hat{f}\, d\mu_\varphi$$

for any $f \in H^\infty(\Delta_\Gamma)$. Hence, $\mu_{F_\infty} = 0$, i.e. $\mu_\varphi = \mu_{\Delta_\Gamma}$. Thus, to each β-continuous linear functional on $H^\infty_\beta(\Delta_\Gamma)$ corresponds some measure $\mu_\varphi \in \mathcal{M}(\Delta_\Gamma)$. □

Corollary. The Shilov β-boundary of the algebra $\partial H^\infty_\beta(\Delta_\Gamma) = \emptyset$.

Proof. According to the local structure of generalized analytic functions mentioned above and due to maximum principle for generalized analytic functions, we have that each generalized ring $K_{\varepsilon,\Gamma} = \{(z,g) \in \mathbb{C}_\Gamma : \varepsilon \leqslant |z| < 1, g \in G\}$ $(0 < \varepsilon < 1)$ is a boundary. But $\bigcap\limits_\varepsilon K_{\varepsilon,\Gamma} = \emptyset$. By Theorem 2 we have, $\partial H^\infty_\beta(\Delta_\Gamma) = \emptyset$, since Δ_Γ is dense in $M_{H^\infty(\Delta_\Gamma)}$. □

In view of Theorems 1, 2 and Corollary we have $M_{H^\infty_\beta(\Delta_\Gamma)} \subsetneq M_{H^\infty(\Delta_\Gamma)}$.

*Proposition.*There exists a linear multiplicative functional $\varphi : H^\infty_\beta(\Delta_\Gamma) \to \mathbb{C}$, which is not continuous in β-topology.

Proof. Δ_Γ is a locally compact space, while $H^\infty(\Delta_\Gamma)$ is a commutative Banach algebra in sup-norm, therefore, Δ_Γ is embedded into $M_{H^\infty(\Delta_\Gamma)}$. On the other hand, Theorem 2 implies $M_{H^\infty_\beta(\Delta\Gamma)} = \Delta_\Gamma$. Let $(\lambda_0, g_0) \in M_{H^\infty(\Delta_\Gamma)} \setminus \Delta_\Gamma$, then the functional $\varphi_{(\lambda_0,g_0)} : H^\infty(\Delta_\Gamma) \to \mathbb{C}$, such that $\varphi_{(\lambda_0,g_0)}(f) = \hat{f}(\lambda_0, g_0)$, is a multiplicative functional on $H^\infty_\beta(\Delta_\Gamma)$. Since $\varphi_{(\lambda_0,g_0)}$ possesses the unique representation measure concentrated in the point (λ_0, g_0), then $\varphi_{(\lambda_0,g_0)}$ is not β-continuous by Theorem 2. □

Thus, every point $(\lambda, g) \in M_{H^\infty(\Delta_\Gamma)} \setminus \Delta_\Gamma$ generates a discontinuous linear multiplicative functional on β-uniform algebra $H^\infty_\beta(\Delta_\Gamma)$.

Concerning the corona problem for the case of β-uniform algebras, we will say that, β-uniform algebra $\mathcal{A}$ has no a corona if $M_{\mathcal{A}_\beta}$ is dense in $*$-weak topology in maximal ideal space $M_{\mathcal{A}_\infty}$.

It is well known that the criterium of non-existence of a corona is the fulfilment of the following condition: for any $\varepsilon > 0$ and a finite family of functions $f_1, \ldots, f_n \in \mathcal{A}$, such that $\inf\limits_{\varphi \in M_{\mathcal{A}_\beta}} \sum\limits_{i=1}^n |\varphi(f_i)| > \varepsilon$, there exist functions $g_1, \ldots, g_n \in \mathcal{A}$ satisfying the equality $\sum\limits_{i=1}^n g_i f_i = 1$.

It is known [3, 4], that Δ_Γ is homeomorphically embedded into the maximal ideal space $M_{H^\infty(\Delta_\Gamma)}$ of the uniform algebra $H^\infty(\Delta_\Gamma)$. Applying the famous Carleson corona theorem, which sets that the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is dense in $M_{H^\infty(\Delta)}$, it is proved in [3] that the "generalized disk" Δ_Γ is dense in $M_{H^\infty(\Delta_\Gamma)}$. In other words, the "corona" $M_{H^\infty(\Delta_\Gamma)} \setminus \bar{\Delta}_\Gamma$ is empty.

In the case of β-uniform algebra $H^\infty_\beta(\Delta_\Gamma)$ we obtain that it also has no a corona, since the set of β-continuous multiplicative linear functionals is Δ_Γ by the Theorem 2, while its Shilov β-boundary $\partial H^\infty_\beta(\Delta_\Gamma) = \emptyset$ (see Corollary).

Observe that, when $\Gamma = \mathbb{Z}$, we get the case of β-uniform algebra $H^\infty_\beta(\Delta)$ on the unit disk Δ.

The author expresses his gratitude to Professor Martin I. Karakhanyan for posing the problem.

Received 15.05.2013

REFERENCES

1. **Gamelin T.** The Uniform Algebras. M.: Mir, 1975 (in Russian).
2. **Grigoryan S.A.** Generalized Analytic Functions. // Uspekhi Matem. Nauk, 1994, v. 49, Issue 2(296), p. 3–42.
3. **Tonev T.V.** The Banach Algebra of Bounded Hyper-Analytic Functions on the Big Disc. // [A] Functions, Series, Operators. Proc. Int. Conf., Budapest 1980, v. 2, Colloq. Math. Soc. Budapest, Hungary, 1980, p. 1189–1194.
4. **Tonev T.V.** The Algebra of Bounded Hyper-Analytic Functions on the Big Disc Has No Corona. // Lecture Notes in Mathematics. Springer-Verlag, 1979, v. 798, p. 435–438.
5. **Hofman K.** Banach Spaces of Analytic Functions. M.: Inostrannaya Literatura, 1975 (in Russian).
6. **Buck R.C.** Bounded Continuous Functions on a Locally Compact Space. // Michigan Math. J., 1958, v. 5, № 2, p. 95–104.
7. **Karakhanyan M.I., Khor'kova T.A.** A Characteristic Property of the Algebra $\mathbb{C}(\Omega)_\beta$. // Sibirsk. Mat. Zh., 2009, v. 50, № 1, p. 96–106; Translation in Sib. Math. J., 2009, v. 50, № 1, p. 77–85.
8. **Gelfand I.M., Raikov D.A., Shilov G.E.** Commutative Normed Rings. M.: Izd. Fiz.-Mat. Lit., 1960 (in Russian).
9. **Rudin W.** Functional Analysis. M.: Mir, 1975 (in Russian).
10. **Grigoryan S.A., Karakhanyan M.I., Khor'kova T.A.** On β-Uniform Dirichlet Algebras. // Journal of Cont. Math. Analysis, 2010, v. 45, № 6, p. 313–319 (in Russian).