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Mathematics

ON A PROPERTY OF GENERAL HAAR SYSTEM

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In the paper we prove that for some type of general Haar systems (particularly for classical Haar system) and for any $\varepsilon > 0$ there exists a set $E \subset (0,1)^2, |E| > 1-\varepsilon$, such that for every $f \in L^1(0,1)^2$ one can find a function $g \in L^1(0,1)^2$, which coincides with f on E and Fourier–Haar coefficients $\{c_{(i,k)}(g)\}_{i,k=1}^{\infty}$ are monotonic over all rays.

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Introduction. In this paper we will consider the behavior of the Fourier–Haar coefficients. Let us recall the definition of the general Haar system.

First we take $t_0 = 0,\ t_1 = 1,\ A_1^{(1)} = \Delta_1 = (0,1),\ h_1(x) = \chi_{(0,1)}$, where χ_E is the characteristic function of the set E.

Then choose any point $t_2 \in (0,1) \setminus \{t_0,t_1\}$, and let $A_1^{(2)} = (0,t_2)$, $A_2^{(2)} = (t_2,1)$, $\Delta_2 = (0,1)$, $\Delta_2^+ = (0,t_2)$, $\Delta_2^- = (t_2,1)$. Define $h_2(x)$ in following manner:

$$h_2(x) = \begin{cases} \frac{1}{2|\Delta_2^+|}, & \text{if } x \in \Delta_2^+, \\ -\frac{1}{2|\Delta_2^-|}, & \text{if } x \in \Delta_2^-. \end{cases}$$

where $|A|$ is the Lebesgue measure of the set A.

In the general case, let the points $t_0,t_1,...,t_n$ be already chosen and let $A_1^{(n)},A_2^{(n)},...,A_n^{(n)}$ be the partition intervals, enumerated from the left to the right, obtained from (0, 1) by removing points $\{t_k\}_{k=1}^n$. Choose any point t_{n+1} from $(0,1) \setminus \{t_0,t_1,...,t_n\}$. Then there exists an interval $A_k^{(n)} = (a,b)$ containing the point t_{n+1}. Denote $\Delta_{n+1} = (a,b)$, $\Delta_{n+1}^+ = (a,t_{n+1})$, $\Delta_{n+1}^- = (t_{n+1},b)$ and

$$h_{n+1}(x) = \begin{cases} \frac{1}{2|\Delta_{n+1}^+|}, & \text{if } x \in \Delta_{n+1}^+, \\ -\frac{1}{2|\Delta_{n+1}^-|}, & \text{if } x \in \Delta_{n+1}^-, \\ 0 & \text{otherwise.} \end{cases}$$

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Note that the value of Haar functions at the points of discontinuity is not essential for the present paper. The only requirement to the points t_n is that the set $\mathcal{T} = \left\{t_k\right\}_{k=0}^{\infty}$ to be dense in [0, 1], i.e.

$$\lim_{n\to+\infty} \max_{1\le k\le n} \left|A_k^{(n)}\right| = 0. \tag{1}$$

The function system $\mathcal{H} = \mathcal{H}_{\mathcal{T}} = \{h_n\}_{n=1}^{\infty}$ is the normalized in $L^1(0,1)$ general Haar system corresponding to the partition $\mathcal{T}$. For each $\mathcal{T}$ (dense in $[0,1]$), the corresponding Haar system $\mathcal{H}$ is a complete orthogonal system in $L^2[0,1]$. There are many papers devoted to the general Haar system, see, e.g., [1,2].

The general Haar system is called an M type system, if for each $m \in \mathbb{N}$ there exists an $n > m$ such that $t_{n+k} \in A_k^{(n)}$, $k = 1,2,...,n$ and

$$\left|A_1^{(2n)}\right| \ge \left|A_2^{(2n)}\right| \ge ... \ge \left|A_{2n}^{(2n)}\right|. \tag{2}$$

For

$$\mathcal{T} = \left\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots\right\},$$

we get the classical Haar system. Note that the classical Haar system is of M type. Let $\mathcal{T}_1,\mathcal{T}_2$ be dense sequences of points in [0,1]. Consider the system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)} = \{h_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^2\}$, where $\mathbf{n} = (n_1,n_2)$, $h_{\mathbf{n}}(x,y) = h^1_{n_1}(x)h^2_{n_2}(y)$ with $h^1_{n_1} \in \mathcal{H}_{\mathcal{T}_1}$, $h^2_{n_2} \in \mathcal{H}_{\mathcal{T}_2}$.

We will say that $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$ is an of M type general Haar system, if $\mathcal{H}_{\mathcal{T}_1}$ and $\mathcal{H}_{\mathcal{T}_2}$ are M type general Haar systems. Note that the Fourier–Haar coefficients $c_{\mathbf{n}}(f,\mathcal{H}_{\mathcal{T}})$, $c_{\mathbf{n}}(f,\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)})$ are determined by formulas

$$c_k(f,\mathcal{H}_{\mathcal{T}}) = \frac{1}{||h_k||_{L^2}} \int_0^1 f(t)h_k(t)dt,$$

$$c_{(i,k)}(f,\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}) = \frac{1}{||h^1_i||_{L^2}||h^2_k||_{L^2}} \int_{(0,1)^2} f(x,y)h^1_i(x)h^2_k(y)dxdy.$$

The spectrum of $f(x,y)$ (denoted by $\underline{spec}(f)$) is the support of $c_{\mathbf{n}}(f)$, i.e. the index set where $c_{\mathbf{n}}(f)$ is non-zero:

$$\underline{spec}(f) = \{\mathbf{n} = (n_1,n_2), c_{\mathbf{n}}(f) \ne 0\}.$$

By $||f||$ we will denote the L^1-norm of f. We will say that the sequence $\{c_{(i,k)}(f)\}_{i,k=1}^{\infty}$ is monotonic over all rays, if

$$|c_{(i_1,k_1)}| \ge |c_{(i_2,k_2)}| \text{ for all } i_1 \ge i_2,\ k_1 \ge k_2 \text{ for which } c_{(i_1,k_1)} \ne 0.$$

The main result of the present paper is the following:

Theorem. For $\forall M$ type Haar system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$, and $\forall \varepsilon > 0$, there exist a set $E \subset (0,1)^2$, $|E| > 1-\varepsilon$ such that for every function $f \in L^1(0,1)^2$ there exists a function $g \in L^1(0,1)^2$, which coincides with f on E and the Fourier–Haar coefficients $\{c_{(i,k)}(g)\}_{i,k=1}^{\infty}$ are monotonic over all rays.

Remark. This result is new for both M type general Haar system and the classical Haar system. Interesting results in this direction were obtained by many mathematicians [3–7].

Auxiliary Lemmas. Let $\mathcal{T}_1,\ \mathcal{T}_2$ be two sequences of points dense in [0,1]. In the rest of the paper we will assume that the M type general Haar system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$ is fixed.

Let $A_1^{(1,n)},A_2^{(1,n)},\ldots,A_n^{(1,n)}\ \left(A_1^{(2,n)},A_2^{(2,n)},\ldots,A_n^{(2,n)}\right)$ be the corresponding partition intervals for $\mathcal{T}_1=\left\{t_k^1\right\}_{k=0}^{\infty}\ \left(\mathcal{T}_2=\left\{t_k^2\right\}_{k=0}^{\infty}\right)$ (2).

L e m m a 1 . For any $N_0\in\mathbb{N}$, $0<\varepsilon<1$, $\delta>0$, $L\neq 0$ and each set $\Delta\in\left\{A_k^{(n)}\right\}$ there exist a polynomial $Q(x)=\sum_{k=N_0}^{N}a_kh_k(x)$ in the M type general Haar system $\mathcal{H}_{\mathcal{T}}$ and a subset $E\subset\Delta$, satisfying the following conditions:

1. $\delta>|a_k|,\ k=N_0,\ N_0+1,\ldots,N;$
2. $|E|>(1-\varepsilon)|\Delta|;\quad ||Q||\leq 2|L||\Delta|;$
3. $Q(x)=\begin{cases}L, & x\in E,\\ 0, & x\notin\Delta;\end{cases}$
4. non-zero coefficients $\{a_k\}_{k\in\underline{spec}(Q)}$ are monotonically decreasing.

Proof. The proof can be done in the same way as the Lemma 2.2 of paper [8], keeping in mind the definition of M type general Haar system (2). □

L e m m a 2 . For any $N_0\in\mathbb{N}$, $\delta>0$, $0<\varepsilon<1$, $L\neq 0$ and each set $\Delta=\Delta_1\times\Delta_2\subset(0,1)^2$, $\Delta_1\in\left\{A_k^{(1,n)}\right\}$, $\Delta_2\in\left\{A_k^{(2,n)}\right\}$ there exist a polynomial

$$Q(x,y)=\sum_{i,k=N_0}^{N}b_{(i,k)}\ h_i^1(x)h_k^2(y)$$

in the M type general Haar system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$ and a subset $E\subset\Delta$ such that:

1. $\delta>|b_{(i,k)}|;\ i,k=N_0,\ N_0+1,\ldots,N;$
2. $|E|>(1-\varepsilon)|\Delta|;\quad ||Q||\leq 4|L||\Delta|;$
3. $Q(x,y)=\begin{cases}L, \text{ if } (x,y)\in E,\\ 0, \text{ if } (x,y)\notin\Delta;\end{cases}$
4. the sequence $\left\{b_{(i,k)}\right\}_{i,k=N_0}^{N}$ is monotonic over all rays.

Proof. By applying Lemma 1 for L, Δ^1 and 1, Δ^2 we will find some sets E^1, E^2 and polynomials $Q^1(x)=\sum_{i=N_0}^{N}a_i^1h_i^1(x)$, $Q^2(y)=\sum_{k=N_0}^{N}a_k^2h_k^2(y)$ satisfying

$$\delta>|a_i^1|,\ 1>|a_k^2|;\ i,k=N_0,\ N_0+1,\ldots,N, \tag{3}$$

$$|E^1|>\left(1-\frac{\varepsilon}{2}\right)|\Delta^1|;\ |E^2|>\left(1-\frac{\varepsilon}{2}\right)|\Delta^2|;$$

$$||Q^1||\leq 2|L||\Delta^1|;\ ||Q^2||\leq 2|\Delta^2|; \tag{4}$$

$$Q^1(x)=\begin{cases}L, & \text{if } x\in E^1,\\ 0, & \text{if } x\notin\Delta^1;\end{cases}\quad Q^2(y)=\begin{cases}1, & \text{if } y\in E^2,\\ 0, & \text{if } y\notin\Delta^2,\end{cases} \tag{5}$$

non-zero coefficients $\left\{a_i^1\right\}_{i\in \underline{spec}(Q^1)}$, $\left\{a_k^2\right\}_{k\in \underline{spec}(Q^2)}$ are monotonically decreasing. We define

$$Q(x,y)=Q^1(x)Q^2(y),\ E=E^1\times E^2. \tag{6}$$

Then it is easy to see that Q and E will satisfy to the conditions of Lemma 2. □

Lemma 3. Let the numbers $N_0>1,\ \varepsilon>0,\ \delta_1>0,\ M$ type general Haar system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$ and the polynomial $f(x,y)=\sum_{r=1}^R L_r\chi_{\Delta_r}$ be given, where $\{\Delta_r\}_{r=1}^R$ are disjoint sets of the form $\Delta_r=\Delta_r^1\times\Delta_r^2$. Then there exist a set $G\subset(0,1)^2$ and a polynomial $Q(x,y)=\sum_{i,k=N_0}^N a_{(i,k)}h_i^1(x)h_k^2(y)$ in the system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$, such that:

1. $\delta>|a_{(i,k)}|;\ i,k=N_0,\ N_0+1,\ldots,N;$
2. $|G|>(1-\varepsilon);\ ||Q||\le 4||f||;$
3. $Q(x,y)=f(x,y)$ for all $(x,y)\in G$;
4. the sequence $\left\{a_{(i,k)}\right\}_{i,k=N_0}^N$ is monotonic over all rays.

Proof. By successively applying Lemma 2, we can find some sets $E_r\subset\Delta_r$ and polynomials

$$Q_r(x,y)=\sum_{i,k=N_{r-1}}^{N_r-1} b_{(i,k)}^r\, h_i^1(x)h_k^2(y),$$

satisfying

$$\delta_r>|b_{(i,k)}^r|;\ i,k=N_{r-1},...,N_r;\ \delta_{r+1}=\min_{i,k,b_{(i,k)}^r\ne 0}|b_{(i,k)}^r|, \tag{7}$$

$$|E_r|>(1-\varepsilon)|\Delta_r|;\qquad ||Q_r||\le 4|L_r||\Delta_r|, \tag{8}$$

$$Q_r(x,y)=\begin{cases}L_r, & \text{if } (x,y)\in E_r,\\ 0, & \text{if } (x,y)\notin\Delta_r,\end{cases} \tag{9}$$

the sequence $\left\{b_{(i,k)}^r\right\}_{i,k=N_{r-1}}^{N_r}$ is monotonic over all rays (for fixed r). We define the set G and the polynomial $Q(x,y)$ in the following manner:

$$G=\bigcup_{r=1}^R E_r,\qquad Q(x,y)=\sum_{r=1}^R Q_r=\sum_{i,k=N_0}^N a_{(i,k)}h_i^1(x)h_k^2(y).$$

It is easy to see that set G and the polynomial $Q(x,y)$ satisfy to the conditions of Lemma 3. □

Proof of the Theorem. Let $\{f_r\}_{r=1}^\infty$ be the sequence of all polynomials with rational coefficients in M type general Haar system $\mathcal{H}_{(\mathcal{T}_1,\mathcal{T}_2)}$.

Successively applying Lemma 3, we will obtain sets $G_r\subset(0,1)^2$ and polynomials

$$Q_r(x,y)=\sum_{i,k=N_{r-1}}^{N_r-1} a_{(i,k)}^r h_i^1(x)h_k^2(y),\ N_0=1,$$

satisfying

$$\delta_r > \left|a^r_{i,k}\right|;\ i,k = N_{r-1},...,N_r,\ \delta_{r+1} = \min_{i,k,a^r_{(i,k)}\neq 0}\left|a^r_{(i,k)}\right|;\ \delta_1 = 1, \tag{10}$$

$$|G_r| > \left(1 - \frac{\varepsilon}{2^r}\right); \quad ||Q_r|| \le 4||f_r||, \tag{11}$$

$$Q_r(x,y) = f_r(x,y) \text{ for all } (x,y) \in G_r \tag{12}$$

the sequence

$$\left\{a^r_{(i,k)}\right\}_{i,k=N_{r-1}}^{N_r-1} \tag{13}$$

is monotonic over all rays.

We denote

$$E = \bigcap_{r=1}^{\infty} G_r. \tag{14}$$

From (11) we have that $|E| > 1-\varepsilon$.

Let $f(x,y)$ be an arbitrary element of $L^1(0,1)^2$. It is easy to see that one can choose a subsequence $\{f_{r_n}\}_{n=1}^{\infty}$ of the sequence $\{f_r\}_{r=1}^{\infty}$ such that

$$\lim_{N\to\infty}\left|\left|\sum_{n=1}^{N} f_{r_n} - f\right|\right| = 0\,, \qquad ||f_{r_n}|| \le 2^{-2n},\ n \ge 2. \tag{15}$$

Define the function $g(x,y)$ as follows;

$$g(x,y) = \sum_{n=1}^{\infty} Q_{r_n}(x,y).$$

From (11)–(15) we get that $g \in L^1(0,1)^2$,

$$g(x,y) = f(x,y) \text{ for } (x,y) \in E$$

and

$$c_{(i,k)}(g) = \frac{1}{||h^1_i||_{L^2}||h^2_k||_{L^2}}\int_{(0,1)^2} g(x,y)h^1_i(x)h^2_k(y)dxdy = a^r_{(i,k)},$$

where $(i,k) \in [N_{r-1}, N_r)^2$.

Then (10) and (13) imply that $\left\{c_{(i,k)}(g)\right\}_{i,k=1}^{\infty}$ is monotonic over all rays. □

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