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Mathematics

ON THE REPRESENTATION OF $\langle\rho_j,\ W_j\rangle$ ABSOLUTE MONOTONE FUNCTIONS

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In the paper [1] the notion of the $\langle\rho_j,\ W_j\rangle$ absolutely monotone function was introduced. In the present paper we give some examples of sequences $\{W_j(x)\}_0^\infty$, consider the corresponding classes of $\langle\rho_j,\ W_j\rangle$ absolute monotone functions and study the problems of thei representation.

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Preliminaries. Let $\alpha \in (0,+\infty)$, $f(x) \in L(0,l)$, $0 < l < +\infty$. The function

$$ {}_0D^{-\alpha} f(x) \equiv D^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,\ x \in (0,l), $$

is called the Riemann-Liouville integral of order α of $f(x)$ with a lower integration limit $x = 0$, and the function

$$ D_l^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^l (t-x)^{\alpha-1} f(t)dt,\ x \in (0,l), $$

is called the Riemann-Liouville integral of function $f(x)$ with upper integration limit $x = l$.

Let $\alpha \in [0,1), 1-\alpha = 1/\rho\ (\rho \geq 1), f(x) \in L(0,l)$. Then the function

$$ D^{1/\rho} f(x) \equiv \frac{d}{dx} D^{-\alpha} f(x) $$

is called the Riemann-Liouville derivative of order $\frac{1}{\rho}$ of $f(x)$ with the initial point $x = 0$. Important properties of these operators are given in [2, 3].

One of these properties is the following: if the functions $f_k(x) \in L(0,l)$, $k = 1,2$, are such that $f_1(x)D_l^{-\alpha} f_2(x) \in L(0,l)$ for some $\alpha \in (0,+\infty)$, then

$$ \int_0^l f_1(x) D_l^{-\alpha} f_2(x) dx = \int_0^l f_2(x) {}_0D^{-\alpha} f_1(x) dx. \tag{1.1} $$

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In [1] the following systems of operators and functions were introduced:

$$A_n^* f(x) \equiv \prod_{j=0}^{n} D_j f(x),\ n \geq 1, D_j f(x) = D^{1/\rho_j}\left\{\frac{f(x)}{W_j(x)}\right\},\ \ j \geq 0; \tag{1.2}$$

$$\tilde{A}_n^* f(x) = D^{-\alpha_n}\left\{\frac{A_{n-1}^* f(x)}{W_n(x)}\right\},\ \ n \geq 0,\ \ A_{-1}^* f(x) \equiv f; \tag{1.3}$$

$$U_0(x) = 1, \quad U_1(x) = \frac{1}{\Gamma(\rho_1^{-1})}\int_0^x \xi_1^{1/\rho_1 - 1} W_1(\xi_1) d\xi_1, \ldots,$$

$$U_n(x) = \frac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_0^x W_1(\xi_1)d\xi_1 \int_0^{\xi_1}(\xi_1 - \xi_2)^{1/\rho_1 - 1} W_2(\xi_2)d\xi_2 \times \cdots \times \tag{1.4}$$

$$\times \int_0^{\xi_n - 1}(\xi_{n-1} - \xi_n)^{1/\rho_{n-1} - 1}\xi_n^{1/\rho_n - 1} W_n(\xi_n) d\xi_n, \quad n \geq 2;$$

$$\Phi_0(t,x) = \begin{cases} 1, & 0 \leq t < x < l, \\ 0, & x \leq t < l, \end{cases}$$

$$\Phi_1(t,x) = \begin{cases} \dfrac{1}{\Gamma(\rho_1^{-1})}\displaystyle\int_t^x (\xi_1 - t)^{\frac{1}{\rho_1} - 1} W_1(\xi_1) d\xi_1, & 0 \leq t < x < l, \\ 0, & x \leq t < l, \end{cases} \tag{1.5}$$

$$\Phi_n(t,x) = \begin{cases} \dfrac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\displaystyle\int_t^x W_1(\xi_1)d\xi_1 \int_t^{\xi_1}(\xi_1 - \xi_2)^{\frac{1}{\rho_1 - 1}} W_2(\xi_2)d\xi_2 \times \cdots \times \\ \times \displaystyle\int_t^{\xi_{n-1}}(\xi_{n-1} - \xi_n)^{\frac{1}{\rho_{n-1}} - 1}(\xi_n - t)^{\frac{1}{\rho_n} - 1} W_n(\xi_n) d\xi_n, & 0 \leq t < x < l, \\ 0, & x \leq t < l. \end{cases} \tag{1.6}$$

We note that for $\rho_j = 1,\ j \geq 0$, these operators and functions were introduced in [4], and for $W_j(x) \equiv 1,\ \rho_j \geq 1, j \geq 0$, they were introduced in [5]. In [1] the following classes of functions were introduced:

a) $C_{n+1}\{[0,l), \langle\rho_j, W_j\rangle\}$ is the set of functions $f(x)$ satisfying the following conditions:

1) the functions $\tilde{A}_k^* f(x),\ k = 0,1,\ldots,n$, are continuous on $[0,l)$;

2) the functions $A_k^* f(x),\ k = 0,1,\ldots,n+1$, are continuous on $(0,l)$ and belong to class $L(0,l)$;

b) $C_\infty\{[0,l),\ \langle\rho_j, W_j\rangle\}$ is the set of functions $f(x)$, which are in the $C_n\{[0,l), \langle\rho_j, W_j\rangle\}$ for any $n \geq 0$;

c) $W = \left\{\{W_j(x)\}_0^\infty | W_0(x) = 1,\ 0 < W_j(x) \in C^\infty[0,l),\ W_j'(x) \leq 0, j \geq 0\right\}$.

It is known (see [1]), that if $f(x) \in C_{n+1}\left\{[0,l),\langle\rho_j,W_j\rangle\right\}$, then the following formula is true:

$$f(x)=\sum_{k=0}^{n}\tilde{A}_k^*f(0)U_k(x)+\int_0^x\Phi_n(t,x)A_n^*f(t)dt. \tag{1.7}$$

The idea of the $\langle\rho_j,W_j\rangle$ absolutely monotonic function was introduced in [1] . We denote the class of such functions by

$$D\left\{[0,l),\langle\rho_j,W_j\rangle\right\}=$$
$$=\left\{f(x)/f(x)\in C_\infty\{[0,l),\langle\rho_j,W_j\rangle\},A_j^*f(x)\geq 0,W_j(x)\in W,j\geq 0\right\}.$$

Note that for $\rho_j=1,W_j(x)=1,j\geq 0$, the class $D\{[0,l),\ \langle 1,1\rangle\}$ is the class of absolutely monotone functions introduced by Bernstein [6]. For $\rho_j=1,\ j\geq 0$, $\left\{W_j(x)\right\}_0^\infty\in W$, the class $D\left\{[0,l),\ \langle 1,W_j\rangle\right\}$ is the class of generalized absolutely monotone functions introduced in [4]. For $\rho_j\geq 1,\ W_j(x)\equiv 1,\ j\geq 0$, the class $D\left\{[0,l),\ \langle\rho_j,1\rangle\right\}$ is the class of absolutely monotone functions introduced in [3]. We note that the idea of the $\langle\rho\rangle$, $\langle\rho,\lambda_j\rangle$ absolutely monotone functions was introduced in [7], but the systems of operators and functions considered there were essentially different.

Assume $\{W_j(x)\}_0^\infty\in W$ satisfies the following property: for any $x_0,\ 0\leq x_0<l$, there exist $x_0^*,\ x_0<x_0^*<l$ and $\varepsilon>0$, such that

$$\lim_{n\to\infty}\prod_{j=1}^{n}\frac{M_j(x_0,x_0^*)}{m_j(x_0,x_0^*)}\varepsilon^{\lambda_n}=0, \tag{1.8}$$

where $m_j(x_0,x_0^*)=\min\limits_{x\in[x_0,x_0^*]}W_j(x)$, $M_j(x_0,x_0^*)=\max\limits_{x\in[x_0,x_0^*]}W_j(x)$, $j=1,\ 2,\ldots$,

$\lambda_n=\sum\limits_{j=1}^{n}\dfrac{1}{\rho_j}$ (obviously, $\lambda_n\to+\infty$, for $n\to\infty$).

In this case we will say that sequence $\left\{W_j(x)\right\}_0^\infty$ has the property "ε". Note that in the case $\rho_j=1,\ j\geq 0,$ the condition (1.8) is given in [4]. Note also that in [1, 8] no examples of $\{W_j(x)\}_0^\infty\in\bar{W}$ are given. In the present paper we will give such examples, we will give a sufficient condition for $\{W_j(x)\}_0^\infty\in\bar{W}$. Also we will give the missing proofs of main lemmas and theorem of [1].

2. Auxiliary Lemmas. Let $\rho_j\geq 1,\ \rho_0=1,\ \alpha_j=1-1/\rho_j,\ j\geq 0,\quad \lambda_0=1$, $0<W_j(x)\in C^\infty[0,l)$. We consider the systems of operators and functions $\left\{A_n^*\right\}_0^\infty$, $\left\{\tilde{A}_n^*\right\}_0^\infty$, $\left\{U_n(x)\right\}_0^\infty$, $\left\{\Phi_n(t,x)\right\}_0^\infty$ defined in (1.2)–(1.6). It is known that for the functions $U_n(x),\ \Phi_n(t,x),\ n\geq 2,$ the following representations are true:

$$U_n(x)=\frac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_0^x\xi_n^{1/\rho_n-1}W_n(\xi_n)d\xi_n\int_{\xi_n}^x(\xi_{n-1}-\xi_n)^{1/\rho_{n-1}-1}W_{n-1}(\xi_{n-1})d\xi_{n-1}\times$$

$$\times\cdots\times\int_{\xi_2}^x(\xi_1-\xi_2)^{1/\rho_1-1}W_1(\xi_1)d\xi_1, \tag{2.1}$$

$$\Phi_n(t,x) = \begin{cases} \dfrac{1}{\prod\limits_{j=1}^{n} \Gamma(\rho_j^{-1})} \displaystyle\int_t^x (\xi_n - t)^{\frac{1}{\rho_n}-1} W_n(\xi_n) d\xi_n \times \\ \quad \times \displaystyle\int_{\xi_n}^x (\xi_{n-1} - \xi_n)^{\frac{1}{\rho_{n-1}}-1} W_{n-1}(\xi_{n-1}) d\xi_{n-1} \times \\ \qquad \times \cdots \times \displaystyle\int_{\xi_2}^x (\xi_1 - \xi_2)^{1/\rho_1 - 1} W_1(\xi_1) d\xi_1, & 0 \le t < x < l, \\ 0, & x \le t < l. \end{cases} \tag{2.2}$$

The following two lemmas can be found in [1, 7].

Lemma 2.1. For any $n \geq 0$:

$$1^\circ.\ A_k^*\{U_n(x)\} \equiv 0,\ k \geq n,\ \tilde{A}_k^*\{U_n(x)\} \equiv 0,\ k \geq n+1,\ x \in [0,l); \tag{2.3}$$

$$2^\circ.\ \tilde{A}_n^*\{U_n(x)\} \equiv 1,\ \forall n \geq 0,\ x \in [0,l); \tag{2.4}$$

$$3^\circ.\ \tilde{A}_k^*\{U_n(x)\}\Big|_{x=0} = 0,\ \forall n \geq 1,\ 0 \leq k \leq n-1. \tag{2.5}$$

Lemma 2.2. The following relations hold:

$$A_k^*\{\Phi_n(t,x)\} \geq 0,\ \tilde{A}_k^*\{\Phi_n(t,x)\} \geq 0,\ k,n = 0,1,\ldots,\ 0 \leq t < x < l. \tag{2.6}$$

Lemma 2.3. If $W_n'(x) \leq 0$, then $\dfrac{d}{dt}\{\Phi_n(t,x)\} \leq 0,\ n \geq 0.$

Proof. Scince by definition, $\Phi_n(t,x) = 0$ for $x \leq t < l$, we have to proof the Lemma for $t \in [0,x)$. Assume $n = 1$. It is easy to see that

$$\Phi_1(t,x) = \frac{1}{\Gamma(\rho_1^{-1})} \int_t^x (\xi_1 - t)^{1/\rho_1 - 1} W_1(\xi_1) d\xi_1 =$$

$$= \frac{\rho_1}{\Gamma(\rho_1^{-1})} \Big((\xi_1 - t)^{\frac{1}{\rho_1}} W_1(\xi_1) \Big|_t^x - \int_t^x (\xi_1 - t)^{1/\rho_1} W_1'(\xi_1) d\xi_1 \Big). \tag{2.7}$$

Then we obtain

$$\frac{d}{dt}\{\Phi_n(t,x)\} = \frac{\rho_1}{\Gamma(\rho_1^{-1})} \Big(-\frac{1}{\rho_1}(x-t)^{\frac{1}{\rho_1}-1} W_1(x) + \frac{1}{\rho_1} \int_t^x (\xi_1 - t)^{\frac{1}{\rho_1}-1} W_1'(\xi_1) d\xi_1 \Big) \leq 0.$$

Now we apply the method of complete induction: assuming the Lemma is true for $n \geq 1$, we will prove that it holds also for $n+1$. Note, that

$$\Phi_{n+1}(t,x) = \frac{\displaystyle\int_t^x (\xi_{n+1} - t)^{\frac{1}{\rho_{n+1}}-1} W_{n+1}(\xi_{n+1}) d\xi_{n+1} \cdots \int_{\xi_2}^x (\xi_1 - \xi_2)^{\frac{1}{\rho_1}-1} W_1(\xi_1) d\xi_1}{\prod\limits_{j=1}^{n+1} \Gamma(\rho_1^{-1})} =$$

$$= \frac{\rho_{n+1}}{\Gamma(\rho_{n+1}^{-1})}\Big((\xi_{n+1}-t)^{\frac{1}{\rho_{n+1}}} W_{n+1}(\xi_{n+1})\Phi_n(\xi_{n+1},x)\Big|_t^x -$$

$$-\int_t^x (\xi_{n+1}-t)^{\frac{1}{\rho_{n+1}}} \frac{dW_{n+1}(\xi_{n+1})\Phi_n(\xi_{n+1},x)}{d\xi_{n+1}} d\xi_{n+1}\Big) = \tag{2.8}$$

$$= -\frac{\rho_{n+1}}{\Gamma(\rho_{n+1}^{-1})}\int_t^x (\xi_{n+1}-t)^{\frac{1}{\rho_{n+1}}} \frac{d}{d\xi_{n+1}}(W_{n+1}(\xi_{n+1})\Phi_n(\xi_{n+1},x))d\xi_{n+1}.$$

From (2.8) we obtain

$$\frac{d(\Phi_{n+1}(t,x))}{dt} = \frac{1}{\Gamma(\rho_{n+1}^{-1})}\int_t^x (\xi_{n+1}-t)^{\frac{1}{\rho_{n+1}}-1}\Big(W'_{n+1}(\xi_{n+1})\Phi_n(\xi_{n+1},x)+$$

$$+W_{n+1}(\xi_{n+1})\frac{d}{d\xi_{n+1}}\Phi_n(\xi_{n+1},x)\Big)d\xi_{n+1} \le 0. \qquad \square$$

Lemma 2.4. The following inequality is true for any $k \ge 0,\ n \ge 0$

$$\tilde{A}_k^*\{U_n(x)\}\Phi_k(x,d) < U_n(d), \qquad 0 \le x < d < l. \tag{2.9}$$

Proof. By Lemma 2.1, $\tilde{A}_k^*\{U_n(x)\} = 0,\ \ \forall n \ge 0,\ k \ge n+1$. So we will prove (2.9) for $0 \le k \le n$. When $k = n$, (2.4) implies that $\tilde{A}_k^*\{U_n(x)\} = 1$. But since $\frac{d}{dt}\{\Phi_n(t,x)\} \le 0$, then from the definition of the function $\Phi_n(x,d)$, it is easy to see that $\Phi_n(x,d) < \Phi_n(0,d) = U_n(d)$. Now let $0 \le k < n-2$, then it is easy to obtain the following formula:

$$\tilde{A}_k^*\{U_n(x)\} =$$
$$= \frac{1}{\prod\limits_{j=k+1}^{n}\Gamma(\rho_j^{-1})}\int_0^x W_{k+1}(\xi_{k+1})d\xi_{k+1}\int_0^{\xi_{k+1}}(\xi_{k+1}-\xi_{k+2})^{\frac{1}{\rho_{k+1}}-1}W_{k+2}(\xi_{k+2})d\xi_{k+2}\times$$

$$\times\cdots\times\int_0^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}\xi_n^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n. \tag{2.10}$$

Consequently for $k = n-2$ and $k = n-1$ we will have

$$\tilde{A}_{n-2}^*\{U_n(x)\} =$$
$$= \frac{1}{\prod\limits_{j=n-1}^{n}\Gamma(\rho_j^{-1})}\int_0^x W_{n-1}(\xi_{n-1})d\xi_{n-1}\int_0^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}\xi_n^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n =$$
$$= \frac{1}{\prod\limits_{j=n-1}^{n}\Gamma(\rho_j^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n\int_{\xi_n}^x(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}W_{n-1}(\xi_{n-1})d\xi_{n-1}, \tag{2.11}$$

$$\tilde{A}^*_{n-1}\{U_n(x)\} = \frac{1}{\Gamma(\rho_n^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)d\xi_n. \tag{2.12}$$

Then by the formula (2.12), using the definition of the function $\Phi_{n-1}(x,d)$ and taking into account (the property) $\frac{d}{dx}\{\Phi_{n-1}(x,d)\} \le 0$, we obtain

$$\tilde{A}^*_{n-1}\{U_n(x)\}\Phi_{n-1}(x,d) = \frac{1}{\Gamma(\rho_n^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)\Phi_{n-1}(x,d)d\xi_n \le$$

$$\le \frac{1}{\Gamma(\rho_n^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)\Phi_{n-1}(\xi_n,d)d\xi_n =$$

$$= \frac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)d\xi_n \int_{\xi_n}^{d}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}W_{n-1}(\xi_{n-1})d\xi_{n-1}\times \tag{2.13}$$

$$\times\ldots\times\int_{\xi_2}^{d}(\xi_1-\xi_2)^{\frac{1}{\rho_1}-1}W_1(\xi_1)d\xi_1 \le U(d).$$

So, we obtain for $0 \le k \le n-2$

$$\tilde{A}^*_k\{U_n(x)\}\Phi_k(x,d) =$$

$$= \frac{1}{\prod\limits_{j=k+1}^{n}\Gamma(\rho_j^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)d\xi_n \int_{\xi_n}^{x}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}W_{n-1}(\xi_{n-1})d\xi_{n-1}\times$$

$$\times\cdots\times\int_{\xi_{k+2}}^{x}(\xi_{k+1}-\xi_{k+2})^{\frac{1}{\rho_{k+1}}-1}W_{k+1}(\xi_{k+1})d\xi_{k+1}\cdot\Phi_k(x,d) < \tag{2.14}$$

$$< \frac{1}{\prod\limits_{j=k+1}^{n}\Gamma(\rho_j^{-1})}\int_0^x \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)d\xi_n\times$$

$$\times\int_{\xi_{k+2}}^{x}(\xi_{k+1}-\xi_{k+2})^{\frac{1}{\rho_{k+1}}-1}W_{k+1}(\xi_{k+1})\Phi_k(\xi_{k+1},d)d\xi_{k+1} <$$

$$< \frac{1}{\prod\limits_{j=k+1}^{n}\Gamma(\rho_j^{-1})}\int_0^d \xi_n^{\frac{1}{\rho_n}-1} W_n(\xi_n)d\xi_n\times$$

$$\times\int_{\xi_{k+2}}^{d}(\xi_{k+1}-\xi_{k+2})^{\frac{1}{\rho_{k+1}}-1}W_{k+1}(\xi_{k+1})\Phi_k(\xi_{k+1},d)d\xi_{k+1}\times$$

$$\times\ldots\times\int_{\xi_1}^{d}(\xi_1-\xi_2)^{\frac{1}{\rho_1}-1}W_1(\xi_1)d\xi_1 = U_n(d). \qquad \square$$

Lemma 2.5. For any $n \geq 1$,

$$\frac{d}{dt}D_x^{-\alpha_n}\{\Phi_n(t,x)\} = -\Phi_{n-1}(t,x)W_n(t), \quad t \neq x. \tag{2.15}$$

Proof. By definition,

$$D_x^{-\alpha_n}\{\Phi_n(t,x)\} = \frac{1}{\Gamma(\alpha_n)}\int_t^x (\tau-t)^{\alpha_n-1}\Phi_n(\tau,x)d\tau =$$

$$= \frac{1}{\alpha_n\Gamma(\alpha_n)}\Big(\Phi_n(\tau,x)(\tau-t)^{\alpha_n}\Big|_t^x - \int_t^x (\tau-t)^{\alpha_n}\frac{d}{d\tau}\{\Phi_n(\tau,x)\}d\tau\Big) = \tag{2.16}$$

$$= -\frac{1}{\alpha_n\Gamma(\alpha_n)}\int_t^x (\tau-t)^{\alpha_n}\frac{d}{d\tau}\{\Phi_n(\tau,x)\}d\tau,$$

so,

$$\frac{d}{dt}D_x^{-\alpha_n}\{\Phi_n(t,x)\} = \frac{1}{\Gamma(\alpha_n)}\int_t^x (\tau-t)^{\alpha_n-1}\frac{d}{d\tau}\{\Phi_n(\tau,x)\}d\tau. \tag{2.17}$$

We note that

$$\Phi_n(\tau,x) = \frac{1}{\prod_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_\tau^x (\xi_n-\tau)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n\times$$

$$\times\int_{\xi_n}^x (\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}W_{n-1}(\xi_{n-1})d\xi_{n-1}\cdots\int_{\xi_2}^x (\xi_1-\xi_2)^{\frac{1}{\rho_1}-1}W_1(\xi_1)d\xi_1 =$$

$$= \frac{\rho_n}{\Gamma(\rho_n^{-1})}\int_\tau^x W_n(\xi_n)\Phi_{n-1}(\xi_n,x)d(\xi_n-\tau)^{\frac{1}{\rho_n}} = \tag{2.18}$$

$$= \frac{\rho_n\big((\xi_n-\tau)^{\frac{1}{\rho_n}}W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\big|_\tau^x - \int_\tau^x (\xi_n-\tau)^{\frac{1}{\rho_n}}\frac{d}{d\xi_n}\{W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\}d\xi_n\big)}{\Gamma(\rho_n^{-1})} =$$

$$= -\frac{\rho_n}{\Gamma(\rho_n^{-1})}\int_\tau^x (\xi_n-\tau)^{\frac{1}{\rho_n}}\frac{d}{d\xi_n}\{W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\}d\xi_n.$$

Than we have

$$\frac{d}{d\tau}\{\Phi_n(\tau,x)\} = \frac{1}{\Gamma(\rho_n^{-1})}\int_\tau^x (\xi_n-\tau)^{\frac{1}{\rho_n}-1}\frac{d}{d\xi_n}\{W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\}d\xi_n. \tag{2.19}$$

Changing the order of integration, from (2.17) and (2.19), we get

$$\frac{d}{dt}D_x^{-\alpha_n}\{\Phi_n(t,x)\} =$$

$$= \frac{1}{\Gamma(\alpha_n)\Gamma(\rho_n^{-1})}\int_t^x (\tau-t)^{\alpha_n-1}dt\int_\tau^x (\xi_n-\tau)^{\frac{1}{\rho_n}-1}\frac{d}{d\xi_n}\{W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\}d\xi_n =$$

$$= \int_t^x \frac{d}{d\xi_n}\big\{\Phi_{n-1}(\xi_n,x)W_n(\xi_n)\big\}d\xi_n = \Phi_{n-1}(\xi_n,x)W_n(\xi_n)\Big|_t^x = -\Phi_{n-1}(t,x)W_n(t). \quad \square$$

L e m m a 2 . 6 . For any $n \geq 0$,

$$\frac{d}{dt}D_x^{-\alpha_n}\big\{\Phi_n(t,x)\big\} = D_x^{-\alpha_n}\Big\{\frac{d}{dt}\Phi_n(t,x)\Big\}, \qquad t \neq x. \tag{2.20}$$

P r o o f . Since, by Lemma 2.5, (2.15) is true, it is sufficient to show that

$$D_x^{-\alpha_n}\Big\{\frac{d}{dt}\Phi_n(t,x)\Big\} = -\Phi_{n-1}(t,x)W_n(t).$$

Using the definition of the operator $D_x^{-\alpha_n}$ and the formula (2.19), we get

$$D_x^{-\alpha_n}\Big\{\frac{d}{dt}\Phi_n(t,x)\Big\} =$$

$$= \frac{1}{\Gamma(\alpha_n)\Gamma(\rho_n^{-1})}\int_t^x (\tau-t)^{\alpha_n-1}d\tau\int_\tau^x(\xi_n-\tau)^{\frac{1}{\rho_n}-1}\frac{d}{d\xi_n}\big\{W_n(\xi_n)\Phi_{n-1}(\xi_n,x)\big\}d\xi_n =$$

$$= \frac{1}{\Gamma(\alpha_n)\Gamma(\rho_n^{-1})}\int_t^x \frac{d}{d\xi_n}\big\{\Phi_{n-1}(\xi_n,x)W_n(\xi_n)\big\}d\xi_n\int_t^{\xi_n}(\tau-t)^{\alpha_n-1}(\xi_n-\tau)^{\frac{1}{\rho_n}-1}d\tau =$$

$$= \frac{1}{\Gamma(\alpha_n)\Gamma(\rho_n^{-1})}\int_t^x \frac{d}{d\xi_n}\big\{\Phi_{n-1}(\xi_n,x)W_n(\xi_n)\big\}\frac{\Gamma(\alpha_n)\Gamma(\rho_n^{-1})}{\Gamma(\alpha_n+\rho_n^{-1})}d\xi_n =$$

$$= \int_t^x \frac{d}{d\xi_n}\big\{\Phi_{n-1}(\xi_n,x)W_n(\xi_n)\big\}d\xi_n = \Phi_{n-1}(\xi_n,x)W_n(\xi_n)\Big|_t^x = -\Phi_{n-1}(t,x)W_n(t). \quad \square$$

In the rest of the paper we will suppose that $\big\{W_j(x)\big\}_0^\infty \in \bar{W}$. Two examples of such sequences are:

a) $W_0(x) \equiv 1,\ W_j(x) = e^{-(\lambda_j-\lambda_{j-1})x},\ \lambda_j = \sum_{k=1}^{j}\frac{1}{\rho_k}, \quad j \geq 1,\ \lambda_0 = 0,\ x \in [0,l).$

It is obvious that $\big\{W_j(x)\big\}_0^\infty \in W$. Clearly, the sequence $\big\{W_j(x)\big\}_0^\infty$ satisfies (1.8), i.e. has the property "ε". Indeed,

$$\prod_{j=1}^n \frac{M_j(x_0,x_0^*)}{m_j(x_0,x_0^*)}\varepsilon^{\lambda_n} = \prod_{j=1}^n \frac{e^{-(\lambda_j-\lambda_{j-1})x_0}}{e^{-(\lambda_j-\lambda_{j-1})x_0^*}}\varepsilon^{\lambda_n} = e^{(x_0^*-x_0)\lambda_n}\varepsilon^{\lambda_n} = e^{\lambda_n(x_0^*-x_0+\ln\varepsilon)} \to 0$$

for $\lambda_n \to +\infty$, if $x_0^*-x_0+\ln\varepsilon < 0,\ 0 < \varepsilon < e^{-(x_0^*-x_0)}$.

b) $W_0(x) \equiv 1,\ W_j(x) = e^{-(\lambda_j^{\frac{1}{q}}-\lambda_{j-1}^{\frac{1}{q}})x},\ j \geq 1$, where q is an integer ($q \geq 2$). It is obvious that $\big\{W_j(x)\big\}_0^\infty \in W$. To show that the sequence $\big\{W_j(x)\big\}_0^\infty \in \bar{W}$, we write

$$\prod_{j=1}^n \frac{M_j(x_0,x_0^*)}{m_j(x_0,x_0^*)}\varepsilon^{\lambda_n} = \prod_{j=1}^n \frac{e^{-(\lambda_j^{\frac{1}{q}}-\lambda_{j-1}^{\frac{1}{q}})x_0}}{e^{-(\lambda_j^{\frac{1}{q}}-\lambda_{j-1}^{\frac{1}{q}})x_0^*}}\varepsilon^{\lambda_n} = e^{(x_0^*-x_0)\lambda_n^{\frac{1}{q}}}\varepsilon^{\lambda_n} = e^{(x_0^*-x_0)\lambda_n^{\frac{1}{q}}+\lambda_n\ln\varepsilon} \to 0$$

for $\lambda_n \to +\infty$, when $0 < \varepsilon < 1$. $\square$

Lemma 2.7. Let $\{W_j(x)\}_0^\infty \in W$. If for any $x_0 (0 \le x_0 < l)$, there exists $x_0^* (x_0 < x_0 < l)$ and $C(x_0, x_0^*)$ such that

$$\prod_{j=1}^{n} \frac{W_j(x_0)}{W_j(x_0^*)} \le C(x_0, x_0^*) \lambda_n, \tag{2.21}$$

then $\{W_j(x)\}_0^\infty \in \bar{W}$.

Proof. Note that $\prod_{j=1}^{n} \frac{M_j(x_0, x_0^*)}{m_j(x_0, x_0^*)} \varepsilon^{\lambda_n} = \prod_{j=1}^{n} \frac{W_j(x_0)}{W_j(x_0^*)} \varepsilon^{\lambda_n} \le C(x_0, x_0^*) \lambda_n \varepsilon^{\lambda_n} \to 0$ for $\lambda_n \to +\infty$ when $0 < \varepsilon < 1 (\lim_{x \to +\infty} x \varepsilon^x = 0)$. □

Next we show that (2.21) is not a necessary condition.

Let $\sup_{j \ge 1} \rho_j = \rho^* < +\infty$, $W_0(x) \equiv 1$, $W_j(x) = e^{-x}$, $j \ge 1$, $\lambda_n = \sum_{j=1}^{n} \frac{1}{\rho_j}$. Then $\frac{n}{\rho^*} \le \lambda_n \le n$. To prove that $\{W_j(x)\}_0^\infty \in \bar{W}$, we calculate

$$\prod_{j=1}^{n} \frac{M_j(x_0, x_0^*)}{m_j(x_0, x_0^*)} \varepsilon^{\lambda_n} = \prod_{j=1}^{n} e^{(x_0^* - x_0)} \varepsilon^{\lambda_n} = e^{n(x_0^* - x_0)} \varepsilon^{\lambda_n} \le e^{n(x_0^* - x_0 + \frac{\ln \varepsilon}{\rho^*})} \to 0$$

for $n \to +\infty$, if $x_0^* - x_0 + \frac{\ln \varepsilon}{\rho^*} < 0$, that is, if $0 < \varepsilon < e^{-\rho^*(x_0^* - x_0)}$.
It can be easily seen that the sequence does not satisfy (2.21). In fact,

$$\prod_{j=1}^{n} \frac{M_j(x_0, x_0^*)}{m_j(x_0, x_0^*)} = e^{n(x_0^* - x_0)}, \text{ but since}$$

$\frac{e^{n(x_0^* - x_0)}}{\lambda_n} \ge \frac{e^{n(x_0^* - x_0)}}{n} \to +\infty$ for $n \to +\infty$, then the condition (2.21) is not satisfied.

It would be interesting to obtain a necessary and sufficient condition for $\{W_j(x)\}_0^\infty \in \bar{W}$. This question has no answer yet.

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