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Mathematics

BOHR'S THEOREM FOR DOUBLE TRIGONOMETRIC INTERPOLATION POLYNOMIALS

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H. Bohr's Theorem on uniform convergence of double trigonometric interpolation polynomials is proved.

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1. Introduction and Main Results. Let $T=[-\pi,\pi]$ and let $C(T)$ $(C(T^2))$ be the space of continuous 2π-periodic functions (in each variable) on $\mathbb{R}$ $(\mathbb{R}^2)$. The following theorem of H. Bohr is well known (see [1]):

Theorem 1.1. For any function $f\in C(T)$ there exists a homeomorphism $\tau(t)$ of T, i.e. a continuous function with

$$-\pi=\tau(-\pi)<\tau(t_1)<\tau(t_2)<\tau(\pi)=\pi,\quad -\pi<t_1<t_2<\pi,$$

such that the Fourier series of the composite function $f\circ\tau(t)$ is uniformly convergent on T.

A stronger version of Bohr's theorem was proved by Kahane and Katznelson in [2] by which an unique homeomorphism τ can be constructed for given compact subset of $C(T)$.

Theorem 1.2 (J.-P. Kahane, Y. Katznelson [2]). Let

$$\omega\in C(0,\infty),\quad 0=\omega(0)<\omega(\delta_1)<\omega(\delta_2)<\infty,\quad 0<\delta_1<\delta_2<\infty. \tag{1}$$

There exists a homeomorphism τ of T such that for any $f\in C(T)$ with modulus of continuity $\omega(\delta,f)<\omega(\delta)$ the Fourier series of the superposition $f\circ\tau(t)$ is uniformly convergent on T.

This Theorem was generalized for multiple Fourier series by Sahakyan [3]. Let $C(T^2)$ be the space of functions, continuous and 2π-periodic in each variable on $\mathbb{R}^2$. For $F\in C(T^2)$ we denote $\omega(\delta,F)=\sup\limits_{(x_1-x_2)^2+(y_1-y_2)^2\le\delta^2}|F(x_1,y_1)-F(x_2,y_2)|$,

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$0 < \delta < \infty$, and let $S_{n,m}(F,x,y)$ be a rectangular partial sum of the Fourier series of F, i.e. $S_{n,m}(F,x,y) = \sum_{k=-n}^{n}\sum_{j=-m}^{m} c_{k,j}(F)e^{i(kx+jy)}, \quad (x,y) \in T^2, \quad n,m = 1,2,\ldots$

Recall that Pringsheim convergence of Fourier series is defined as the convergence of rectangular partial sums.

Theorem 1.3 [3]. For any function $\omega(\delta)$, which satisfies (1), there exists a homeomorphism τ of T such that the Fourier series of arbitrary F in the form

$$F(x,y) = f(\tau(x),\tau(y)), \quad f \in C(T^2), \quad \omega(\delta, f) \le \omega(\delta) \tag{2}$$

is Pringsheim uniformly convegent on T^2. Moreover, for an arbitrary ε there exists a number N such that $\|S_{n,m}(F) - F\|_C \le \varepsilon$, $n,m > N$, for any function of the form (2).

Hereafter notations of Sahakyan [3] will be used. Let $M(T)$ $(M(T^2))$ be the space of 2π-periodic (in each variable), measurable and bounded functions on $\mathbb{R}$ $(\mathbb{R}^2)$, and $\|f\|_\infty = \sup|f(x)|$. We define intervals Δ_k^δ as follows

$$\Delta_k^\delta(x,n) := \left(x+\delta\frac{2k-1}{n}\pi, x+\delta\frac{2k}{n}\pi\right), \quad x \in T, \quad k,n = 1,2,\ldots, \quad \delta = \pm 1.$$

For $\Delta = (a,b)$ and $f \in M(T)$ denote $f(\Delta) = f(b) - f(a)$ and

$$W_n^\delta(f,x) := \left|\sum_{k=1}^{[n/2]} \frac{f(\Delta_k^\delta(x,n))}{k}\right|, \qquad x \in T, \quad n = 1,2,\ldots, \quad \delta = \pm 1,$$

$$W_n(f) := \sup_{x\in T,\ \delta=\pm 1} W_n^\delta(f,x), \quad n = 1,2,\ldots$$

For one-dimensional Fourier series the Salem's test is well known.

Theorem 1.4 [1]. If $f \in C(T)$ and

$$\lim_{n\to\infty} W_n^\delta(f) = 0, \quad \delta = \pm 1,$$

then the Fourier series of function f is uniformly convergent on T.

In [4] Golubov has proved the analogue of Salem's theorem for 2-dimensional case. To state the theorem we need some more definitions for 2-dimensional case.

Let $F \in M(T^2)$ and $F(\Delta_1,\Delta_2) := F(x_1,y_1) + F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1)$, where $\Delta_1 = (x_1,y_1), \Delta_2 = (x_2,y_2)$. Then define

$$W_{n,m}^{\delta_1,\delta_2}(F,x,y) := \left|\sum_{k=1}^{[n/2]}\sum_{j=1}^{[m/2]} \frac{F(\Delta_k^{\delta_1}(x,n), \Delta_j^{\delta_2}(y,m))}{kj}\right|,$$

where $(x,y) \in T^2, \quad n,m = 1,2,\ldots, \quad \delta_1,\delta_2 = \pm 1$. Analogously, as in 1-dimensional case, we define

$$W_{n,m}(F) := \sup_{(x,y)\in T^2,\ \delta_1,\delta_2=\pm 1} \{W_{n,m}^{\delta_1,\delta_2}(F,x,y) + W_n^{\delta_1}((F,\cdot,y),x) + W_m^{\delta_2}((F,x,\cdot),y)\}.$$

Theorem 1.5 (Golubov, [4]). Fourier series of functions $F \in C(T^2)$, for which $\lim_{n,m\to\infty} W_{n,m}(F) = 0$, is Pringsheim uniformly convergent on T^2.

Remark 1.1. As noted in [3], analogue of Theorem 1.3 for multidimensional case holds and homeomorphism τ can be constructed independently from dimension.

For $F \in M(T^2)$, $\quad x,y,a,b \in T$ denote

$$V_{a,b}(x,y) := \sum_{k=1}^{\left[\frac{\pi}{a}\right]} \sum_{j=1}^{\left[\frac{\pi}{b}\right]} \frac{|F(I_k,\Delta_j)|}{kj} + \sum_{k=1}^{\left[\frac{\pi}{a}\right]} \frac{|F(I_k,y)|}{k} + \sum_{j=1}^{\left[\frac{\pi}{b}\right]} \frac{|F(x,\Delta_j)|}{j},$$

where $I_k = (x+(k-1)a, x+ka)$, $\quad \Delta_j = (y+(j-1)b, y+jb)$. Next, denote

$$V_h(F) := \sup_{|a|,|b|\le h,(x,y)\in T^2} V_{a,b}(x,y), \quad V(F) := \sup_{h>0} V_h(F).$$

It's easy to see, that if $\max\left(\frac{\pi}{n},\frac{\pi}{m}\right) \le h$, then $W_{n,m}(F) \le V_h(F)$. This means that the result of Theorem 1.5 will hold when $\lim\limits_{h\to 0} V_h(F) = 0$.

For $t_0, s_0 \in T$, naturals N, M, we define $h_N = \dfrac{2\pi}{2N+1}$, $\quad h_M = \dfrac{2\pi}{2M+1}$ and

$$t_i = t_i^N = t_0 + ih_N, \quad s_j = s_j^M = s_0 + jh_M, \quad i,j = 0,\pm 1,\dots \tag{3}$$

For given 2π-periodic in each variable function $F(x,y)$, $(x,y) \in \mathbb{R}^2$, we denote by $I_{N,M}(F,x,y)$ the unique trigonometric polynomial with degrees of N,M in x,y respectively:

$$I_{N,M}(F,x,y) = \sum_{\nu=-N}^{N} \sum_{\mu=-M}^{M} c_{\nu,\mu}^{N,M} e^{i\nu x} e^{i\mu y},$$

which coincides with F at the points (t_i, s_j), i.e.

$$I_{N,M}(F,t_i,s_j) = F(t_i,s_j), \quad i,j = 0,\pm 1,\dots$$

Partial sums of $I_{N,M}(F)$ for $n = 0,1,\dots,N$, $m = 0,1,\dots,M$ are defined as follows:

$$\begin{aligned} I_{n,m}^{N,M}(F,x,y) &= \sum_{\nu=-n}^{n} \sum_{\mu=-m}^{m} c_{\nu,\mu}^{N,M} e^{i\nu x} e^{i\mu y} = \\ &= \frac{1}{\pi^2} \int_{T^2} F(t,s) D_n(x-t) D_m(y-s)\, d\omega_{2N+1}(t)\, d\omega_{2M+1}(s) = \qquad (4) \\ &= \frac{1}{\pi^2} \int_{T^2} F(x+t,y+s) D_n(t) D_m(s)\, d\widetilde{\omega}_{2N+1}(t)\, d\widetilde{\omega}_{2M+1}(s), \end{aligned}$$

where $\omega_{2N+1}(t)$ $(\omega_{2M+1}(s))$ is a left continuous step-function having jumps h_N (h_M) at the points t_i (s_j), and $\widetilde{\omega}_{2N+1}(t) = \omega_{2N+1}(x+t)$, $\quad \widetilde{\omega}_{2M+1}(s) = \omega_{2M+1}(y+s)$.

The main results of this paper are the following theorems.

Theorem 1.6. If $F \in C(T^2)$ and $\lim\limits_{n,m\to\infty} W_{n,m}(F) = 0$, then the polynomials $I_{n,m}^{N,M}(F)$ are uniformly convergent on T^2 to F, as $n,m \to \infty$, $n \le N$, $m \le M$.

Theorem 1.7. For any function $\omega(\delta)$ in the form (1) there exists a homeomorphism τ of T such that for any function F in the form

$$F(x,y) = f(\tau(x),\tau(y)), \quad f \in C(T^2), \quad \omega(\delta,f) \le \omega(\delta), \tag{5}$$

the polynomials $I_{n,m}^{N,M}(F)$ are uniformly convergent on T^2 to F, as $n,m \to \infty$, $n \le N$, $m \le M$. Moreover, for an arbitrary $\varepsilon > 0$ there exists a number K such that $\|I_{n,m}^{N,M}(F) - F\|_C \le \varepsilon$, $\quad n,m > k$, for any function F in the form (5).

2. Auxiliary Results.

Lemma 2.1. For any function $\phi \in M(T)$

$$\frac{1}{\pi}\int_T \phi(x+t)\frac{\sin nt}{t}d\omega_{2N+1}(t) = \phi(x) + O(U_n(\phi)),$$

where

$$U_n(\phi) := W_n(\phi) + \omega\left(\frac{\ln n}{n}, \phi\right) + o(1)\cdot\|\phi\|_\infty,$$

and the quantity $o(1)$ depends on n, x, N and tends to 0 as $n \to \infty$ uniformly in $x \in T$ and $N > n$.

Proof. It is well known (see [5], Ch. X) for one dimensional trigonometric interpolation polynomials $I_n^N(f)$ that $\lim\limits_{n\to\infty} I_n^N(f,x) = f(x)$, if f is of bounded variation and x is a point of continuity. On the other hand, by Theorem 1 in [6],

$$I_n^N(f,x) = \frac{1}{\pi}\int_T f(x+t)\frac{\sin nt}{t}d\omega_{2N+1}(t) + o(1).$$

Obviously, setting $f \equiv 1$ yields

$$\frac{1}{\pi}\int_T \phi(x+t)\frac{\sin nt}{t}d\omega_{2N+1}(t) - \phi(x) =$$
$$= \frac{1}{\pi}\int_T \left[\phi(x+t) - \phi(x)\right]\frac{\sin nt}{t}d\omega_{2N+1}(t) + o(1)\cdot\phi(x).$$

It is enough to estimate the integral

$$I := \frac{1}{\pi}\int_0^\pi \left[\phi(x+t) - \phi(x)\right]\frac{\sin nt}{t}d\omega_{2N+1}(t).$$

Denote $\psi(t) := \phi(x+t) - \phi(x)$ and observe that

$$|\psi(t)| \le \omega(t,\phi), \quad 0 < t \le \pi.$$

Hence,

$$I = \frac{1}{\pi}\int_0^\pi \psi(t)\frac{\sin nt}{t}d\omega_{2N+1}(t) =$$
$$= \frac{1}{\pi}\int_0^{h_n} \psi(t)\frac{\sin nt}{t}d\omega_{2N+1}(t) + \frac{1}{\pi}\int_{h_n}^\pi \psi(t)\frac{\sin nt}{t}d\omega_{2N+1}(t) =: I_1 + I_2.$$

The first integral can be easily estimated as follows:

$$I_1 \le \frac{1}{\pi}\left(\frac{h_n}{h_N} + 1\right)\omega(h_n,\phi)nh_N \le 2\pi\omega(h_n,\phi).$$

For the second integral we need more notations. Put

$$p_1 = \left[\frac{h_n}{h_N}\right], \quad q_1 = \left[\frac{\pi}{p_1 h_N}\right], \quad p_2 = \left[\frac{h_m}{h_M}\right], \quad q_2 = \left[\frac{\pi}{p_2 h_M}\right],$$

where $[a]$ is the integer part of a. Denote

$$\tau_i^j = t_0 + (jp_1 + i)h_N, \quad i = 1,2,\dots,p_1, \quad j = 0,1,\dots \tag{6}$$
$$\sigma_k^l = s_0 + (lp_2 + k)h_M, \quad k = 1,2,\dots,p_2, \quad l = 0,1,\dots$$

To estimate I_2, we divide the interval $[h_n, \pi]$ into intervals $[h_n, q_1 p_1 h_N]$ and $[q_1 p_1 h_N, \pi]$. The integral on the second interval can be estimated in the exact same way as the one on $[0, h_n]$. Hence, we need to estimate the integral on $[h_n, q_1 p_1 h_N]$, which can be represented as a finite sum as follows:

$$\int_{h_n}^{q_1 p_1 h_N} \psi(t) \frac{\sin nt}{t} d\omega_{2N+1}(t) = \sum_{j=1}^{q_1-1} \sum_{i=1}^{p_1} \psi(\tau_i^j) \frac{\sin n\tau_i^j}{\tau_i^j} h_N. \quad (7)$$

We fix i and proceed with the sum across j. Applying the Abel summation formula, we get

$$\sum_{j=1}^{q_1-1} \psi(\tau_i^j) \frac{\sin n\tau_i^j}{\tau_i^j} h_N =$$

$$= \sum_{j=1}^{q_1-2} \left[\frac{\psi(\tau_i^j)}{\tau_i^j} - \frac{\psi(\tau_i^{j+1})}{\tau_i^{j+1}} \right] Q_i^j h_N + \frac{\psi(\tau_i^{q_1-1})}{\tau_i^{q_1-1}} Q_i^{q_1-1} h_N = J_1 + J_2, \quad (8)$$

where

$$Q_i^j = \sum_{r=1}^{j} \sin n\tau_i^r.$$

In [6] (p. 551) it is proved that $|Q_i^j| < 2$. Hence,

$$J_2 \le \frac{\|\psi\|_\infty}{(q_1-1)p_1 h_N} 2h_N \quad \text{and} \quad p_1 J_2 \le 2 \frac{\|\psi\|_\infty}{n-1} \quad (9)$$

as

$$q_1 - 1 = \left[\frac{\pi}{p_1 h_N} \right] - 1 \ge \left[\frac{\pi}{h_n} \right] - 1 = \left[\frac{2n+1}{2} \right] = n - 1.$$

We divide J_1 into two parts as follows:

$$J_1 = \sum_{j=1}^{q_1-2} \left[\frac{\psi(\tau_i^j)}{\tau_i^j} - \frac{\psi(\tau_i^{j+1})}{\tau_i^{j+1}} \right] Q_i^j h_N = \sum_{j=1}^{q_1-2} \left[\frac{\psi(\tau_i^j) - \psi(\tau_i^{j+1})}{\tau_i^j} \right] Q_i^j h_N +$$

$$+ \sum_{j=1}^{q_1-2} \psi(\tau_i^{j+1}) \left[\frac{1}{\tau_i^j} - \frac{1}{\tau_i^{j+1}} \right] Q_i^j h_N = J_{1,1} + J_{1,2}. \quad (10)$$

The estimation of $J_{1,1}$ is straightforward:

$$|J_{1,1}| \le \sum_{j=1}^{q_1-2} \frac{|\psi(\tau_i^j) - \psi(\tau_i^{j+1})|}{p_1 j h_N} |Q_i^j| h_N \quad \text{and} \quad p_1 |J_{1,1}| = O(W_n(\phi)). \quad (11)$$

For $J_{1,2}$ we have

$$|p_1 J_{1,2}| = p_1 \left| \sum_{j=1}^{q_1-2} \frac{\psi(\tau_i^{j+1}) p_1 h_N}{j(j+1) p_1 h_N p_1 h_N} Q_i^j h_N \right| \le 2 \sum_{j=1}^{q_1-2} \frac{|\psi(\tau_i^{j+1})|}{j(j+1)} \le 2 \sum_{j=1}^{[\ln n]} \frac{|\psi(\tau_i^{j+1})|}{j(j+1)} +$$

$$+ 2 \sum_{j=[\ln n]+1}^{\infty} \frac{|\psi(\tau_i^{j+1})|}{j(j+1)} \le C\omega(p_1 h_N \ln n, \phi) + o(1) \cdot \|\psi\|_\infty. \quad (12)$$

From (8)–(12) we get the desired estimate for the integral in (7). □

Lemma 2.2. For any function $f \in M(T)$

$$c_n^N(f) := \frac{1}{2\pi}\int_T f(t)e^{-int}d\omega_{2N+1}(t) \le C\left(\omega\left(f,\frac{\pi}{n}\right)+\frac{\|f\|_\infty}{n}\right),$$

where C is an absolute constant.

Proof. Technique of the proof is very similar to the one in previous Lemma. We divide the interval $[0,\pi]$ into three subintervals $[0,h_n]$, $[h_n, p_1q_1h_N]$ and $[p_1q_1h_N,\pi]$ (the integral over the interval $[-\pi,0]$ can be estimated similarly). Integrals on the small intervals are easily estimated. We have

$$\frac{1}{2\pi}\int_0^{h_n} f(t)e^{-int}d\omega_{2N+1}(t) \le \frac{h_N}{2\pi}\left(\frac{h_n}{h_N}+1\right)\|f\|_\infty \le \frac{\|f\|_\infty}{n}.$$

The integral on the third interval is estimated in the exact same way. We now turn to the integral on interval $[h_n, p_1q_1h_N]$. As in the previous Lemma we write this integral as a sum:

$$G := \int_{h_n}^{p_1q_1h_N} f(t)e^{-int}d\omega_{2N+1}(t) = h_N\sum_{j=1}^{q_1-1}\sum_{r=1}^{p_1} f(\tau_r^j)e^{-in\tau_r^j}.$$

Again, we fix r and consider the sum over $j = 1,\dots,q_1-1$. Applying the Abel summation formula, we get:

$$h_N\sum_{j=1}^{q_1-1} f(\tau_r^j)e^{-in\tau_r^j} = h_N\sum_{j=1}^{q_1-2}[f(\tau_r^j)-f(\tau_r^{j+1})]A_r^j + h_N f(\tau_r^{q_1-1})A_r^{q_1-1},$$

where $A_r^j = \sum_{k=1}^{j} e^{-in\tau_r^k}$. Notice that $|A_r^j| = \dfrac{|e^{-inp_1h_N(j+1)}-1|}{|e^{-inp_1h_N}-1|} = \dfrac{1}{2\sin\left(\frac{np_1h_N}{2}\right)} \le 1$,

because

$$\frac{\pi}{2} \ge \frac{np_1h_N}{2} \ge n\frac{p_1}{2(p_1+1)}(p_1+1)h_N \ge n\frac{p_1}{2(p_1+1)}h_n = \frac{p_1}{p_1+1}\cdot\frac{n}{2n+1}\pi \ge \frac{\pi}{6}.$$

Consequently, $$G \le \left|p_1h_N\sum_{j=1}^{q_1-1} f(\tau_r^j)e^{-in\tau_r^j}\right| \le \left|p_1h_N\sum_{j=1}^{q_1-2}[f(\tau_r^j)-f(\tau_r^{j+1})]A_r^j\right| +$$

$$+\left|p_1h_Nf(\tau_r^{q_1-1})A_r^{q_1-1}\right| \le C\left(\omega\left(f,\frac{\pi}{n}\right)+\frac{\|f\|_\infty}{n}\right). \quad \square$$

Denote

$$g(t) := \frac{1}{2\tan\frac{t}{2}} - \frac{1}{t}, \quad 0<|t|\le\pi, \quad g(0)=0.$$

Lemma 2.3. There exists an absolute constant $K>0$ such that for any function $F \in M(T^2)$ and $(x,y)\in T^2$

$$\left|\int_{T^2} F(x+t,y+s)g(s)\frac{\sin nt}{t}e^{ims}d\omega_{2N+1}(t)d\omega_{2M+1}(s)\right| \le K\cdot U_{n,m}(F),$$

where

$$U_{n,m}(F) := W_{n,m}(F) + \omega\left(\frac{\ln n}{n}+\frac{\ln m}{m},F\right) + \gamma_{n,m}\|F\|_\infty$$

and $\gamma_{n,m} \to 0$ as $n,m \to \infty$ uniformly on $(x,y) \in T^2$, $N > n$, $M > m$.

P r o o f. The proof of the Lemma follows from Lemmas 2.1 and 2.2. Using Lemma 2.1, we get

$$\int_T F(x+t,y+s)\frac{\sin nt}{t}d\omega_{2N+1}(t) = F(x,y+s) + O(U_n(F(\cdot,y+s))).$$

On the other hand,

$$U_n(F(\cdot,y+s)) = W_n(F(\cdot,y+s)) + \omega\left(\frac{\ln n}{n}, F(\cdot,y+s)\right) + \|F(\cdot,y+s)\|_\infty \eta_n,$$

where η_n depends on n,x,N and tends to 0 as $n \to \infty$ uniformly in $x \in T$ and $N > n$ as shown in Lemma . Hence,

$$U_n(F(\cdot,y+s)) \leq W_{n,m}(F) + \omega\left(\frac{\ln n}{n} + \frac{\ln m}{m}, F\right) + \|F\|_\infty \eta_n \leq U_{n,m}(F).$$

Using the previous inequality, we obtain

$$\int_T F(x+t,y+s)\frac{\sin nt}{t}d\omega_{2N+1}(t) = F(x,y+s) + O(U_{n,m}(F)).$$

So, we need to estimate the Fourier coefficients of the function $u(s) := F(x,y+s)g(s)$. We have

$$\begin{aligned} |u(s+h)-u(s)| &\leq |F(x,y+s+h)-F(x,y+s)|\cdot|g(s+h)| + \\ &\quad +|F(x,y+s)|\cdot|g(s+h)-g(s)| \leq \\ &\leq \|g\|_\infty \cdot \omega(h,F) + \|F\|_\infty \cdot |g(s+h)-g(s)|. \end{aligned}$$

Hence, Lemma 2.2 yields

$$c_m^M(u) \leq C\left[\omega\left(u,\frac{\pi}{m}\right) + \frac{\|F\|_\infty \cdot \|g\|_\infty}{m}\right] \leq C_1\left[\omega\left(F,\frac{\pi}{m}\right) + \frac{\|F\|_\infty}{m} + \frac{1}{m}\right].$$

The absolute value of the integral $\int_T O(U_{n,m}(F))g(s)e^{ims}d\omega_{2M+1}(s)$ is estimated straightforwardly as g is bounded on T and $\|e^{ims}\| = 1$. The result of the lemma follows from the last two estimates. □

R e m a r k 2 . 1 . It is easy to see that the Lemma holds, if $g \equiv 1$.

3. Proof of the Main Result.

L e m m a 3 . 1 . For any $F \in M(T^2)$ and any node set (3),

$$I_{n,m}^{N,M}(F,x,y) = \frac{1}{\pi^2}\int_T F(x+t,y+s)g(s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s) + \\ + O(U_{n,m}(F)).$$

P r o o f. We have that

$$I_{n,m}^{N,M}(F,x) = \frac{1}{\pi^2}\int_{T^2} F(x+t,y+s)D_n(t)D_m(s)d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s),$$

where $D_N(t) = \dfrac{\sin(N+\frac{1}{2})t}{2\sin\frac{1}{2}t} = \dfrac{\sin Nt}{t} + g(t)\sin nt + \dfrac{1}{2}\cos nt.$

Consequently,

$$\pi^2 I_{N,M}^{n,m}(f,x,y) = \int_{T^2} F(x+t,y+s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)+$$
$$+\int_{T^2} F(x+t,y+s)\frac{\sin nt}{t}\left[g(s)\sin ms+\frac{1}{2}\cos ms\right]d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)+$$
$$+\int_{T^2} F(x+t,y+s)\frac{\sin ms}{s}\left[g(t)\sin nt+\frac{1}{2}\cos nt\right]d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)+$$
$$+\int_{T^2} F(x+t,y+s)\left[g(t)\ \sin nt\ +\ \frac{1}{2}\cos nt\right]\times$$
$$\times\left[g(s)\sin ms+\frac{1}{2}\cos ms\right]d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)=$$
$$=\int_{T^2} F(x+t,y+s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)+\sum_{p=1}^{3} I_{N,M}^{n,m,p}(x,y),$$

and the result follows using Lemma 2.3 for $p=1,2$ and Lemma 2.2 for $p=3$. □

Lemma 3.2. For any $F\in M(T^2)$ and any node set (3),

$$\frac{1}{\pi^2}\int_{T^2} F(x+t,y+s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\widetilde{\omega}_{2N+1}(t)d\widetilde{\omega}_{2M+1}(s)=F(x,y)+O(U_{n,m}(F)).$$

Proof. Obviously it is sufficient to prove the Lemma in the positive quadrant. Denoting $\phi(t,s)=F(x+t,y+s)-F(x,y)$, we have

$$\int_0^{\pi}\int_0^{\pi}\phi(t,s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\omega_{2N+1}(t)d\omega_{2M+1}(s)=\int_0^{h_m}\int_0^{h_n}+\int_{h_m}^{\pi}\int_0^{h_n}+\int_0^{h_m}\int_{h_n}^{\pi}+$$
$$+\int_{q_2p_2h_M}^{\pi}\int_{h_n}^{\pi}+\int_{h_m}^{q_2p_2h_M}\int_{q_1p_1h_N}^{\pi}+\int_{h_m}^{q_2p_2h_M}\int_{h_n}^{q_1p_1h_N}=\sum_{k=1}^{6}I_k. \tag{13}$$

We estimate the integrals I_k seperately. For $p=1$ we have

$$|I_1| \le \left(\frac{h_n}{h_N}+1\right)\left(\frac{h_m}{h_M}+1\right)nmh_Nh_M\omega\left(F,\sqrt{h_n^2+h_m^2}\right)\le$$
$$\le C\omega\left(F,\sqrt{h_n^2+h_m^2}\right)=O(U_{n,m}(F)). \tag{14}$$

For $p=2$, using Lemma 2.1, we have

$$|I_2| = \left|\int_{h_m}^{\pi}\int_0^{h_n}\phi(t,s)\frac{\sin nt}{t}\cdot\frac{\sin ms}{s}d\omega_{2N+1}(t)d\omega_{2M+1}(s)\right|=$$
$$=\left|\int_0^{h_n}\frac{\sin nt}{t}\int_{h_m}^{\pi}\phi(t,s)\frac{\sin ms}{s}d\omega_{2M+1}(s)d\omega_{2N+1}(t)\right|\le$$
$$\le\left|\int_0^{h_n}\frac{\sin nt}{t}O[U_m(F(x+t,y+\cdot))]d\omega_{2N+1}(t)\right|\le \tag{15}$$
$$\le\left(\frac{h_n}{h_N}+1\right)nh_NO(U_{n,m}(F))=O(U_{n,m}(F)).$$

The proof for $p=3$ is exactly the same as in the case $p=2$. When $p=4$ we have (the case $p=5$ is similar)

$$\begin{aligned} |I_4| &= \left| \int_{\sigma_{p_2}^{q_2-1}}^{\pi} \int_{h_n}^{\pi} \phi(t,s) \frac{\sin nt}{t} \cdot \frac{\sin ms}{s} d\omega_{2N+1}(t) d\omega_{2M+1}(s) \right| \leq \\ &\leq \left| \int_{\sigma_{p_2}^{q_2-1}}^{\pi} \frac{\sin ms}{s} O[U_m(F(x+\cdot, y+s)] d\omega_{2M+1}(s) \right| \leq \\ &\leq p_2 h_M \frac{1}{q_2 p_2 h_M} O(U_{n,m}(F)) = O(U_{n,m}(F)). \end{aligned} \tag{16}$$

In the case $p=6$ we can write

$$I_6 = h_N h_M \sum_{i=1}^{p_1} \sum_{k=1}^{p_2} \sum_{j=1}^{q_1-1} \sum_{l=1}^{q_2-1} \frac{\phi(\tau_i^j, \sigma_k^l)}{\tau_i^j \sigma_k^l} \sin n\tau_i^j \sin m\sigma_k^l, \tag{17}$$

as the first node greater than h_n is the $(p_1 h_N + h_N)$, which is, by definition, equal to τ_1^1 (same argument applies to h_m), and the last node in the integral of $p=6$ is the $p_1 q_1 h_N$, which is, by definition, equal to $\tau_{p_1}^{q_1-1}$ (same argument applies to $p_2 q_2 h_M$). Note that the nodes $p_1 q_1 h_N$ and $p_2 q_2 h_M$ are included in the integrals over the intervals $[h_n, p_1 q_1 h_N]$ and $[h_m, p_2 q_2 h_M]$. Keeping i,k fixed, we estimate the following sum:

$$J := \sum_{j=1}^{q_1-1} \sum_{l=1}^{q_2-1} \frac{\phi(\tau_i^j, \sigma_k^l)}{\tau_i^j \sigma_k^l} \sin n\tau_i^j \sin m\sigma_k^l. \tag{18}$$

Using Abel summation formula, we get

$$\begin{aligned} J &= \sum_{j=1}^{q_1-1} \sum_{l=1}^{q_2-1} \frac{\phi(\tau_i^j, \sigma_k^l)}{\tau_i^j \sigma_k^l} \sin n\tau_i^j \sin m\sigma_k^l = \sum_{j=1}^{q_1-1} \frac{\sin n\tau_i^j}{\tau_i^j} \sum_{l=1}^{q_2-1} \frac{\phi(\tau_i^j, \sigma_k^l)}{\sigma_k^l} \sin m\sigma_k^l = \\ &= \sum_{j=1}^{q_1-1} \frac{\sin n\tau_i^j}{\tau_i^j} \sum_{l=1}^{q_2-2} \left[\frac{\phi(\tau_i^j, \sigma_k^l)}{\sigma_k^l} - \frac{\phi(\tau_i^j, \sigma_k^{l+1})}{\sigma_k^{l+1}} \right] P_k^l + \sum_{j=1}^{q_1-1} \frac{\sin n\tau_i^j}{\tau_i^j} \frac{\phi(\tau_i^j, \sigma_k^{q_2-1})}{\sigma_k^{q_2-1}} P_k^{q_2-1} = \\ &= \sum_{l=1}^{q_2-2} P_k^l \sum_{j=1}^{q_1-1} \left[\frac{\phi(\tau_i^j, \sigma_k^l)}{\tau_i^j \sigma_k^l} - \frac{\phi(\tau_i^j, \sigma_k^{l+1})}{\tau_i^j \sigma_k^{l+1}} \right] \sin n\tau_i^j + P_k^{q_2-1} \sum_{j=1}^{q_1-1} \frac{\phi(\tau_i^j, \sigma_k^{q_2-1})}{\tau_i^j \sigma_k^{q_2-1}} \sin n\tau_i^j = \\ &= \sum_{l=1}^{q_2-2} P_k^l \sum_{j=1}^{q_1-2} \left[\frac{\phi(\tau_i^j, \sigma_k^l)}{\tau_i^j \sigma_k^l} - \frac{\phi(\tau_i^j, \sigma_k^{l+1})}{\tau_i^j \sigma_k^{l+1}} - \frac{\phi(\tau_i^{j+1}, \sigma_k^l)}{\tau_i^{j+1} \sigma_k^l} + \frac{\phi(\tau_i^{j+1}, \sigma_k^{l+1})}{\tau_i^{j+1} \sigma_k^{l+1}} \right] Q_i^j + \\ &+ \sum_{l=1}^{q_2-2} P_k^l \left[\frac{\phi(\tau_i^{q_1-1}, \sigma_k^l)}{\tau_i^{q_1-1} \sigma_k^l} - \frac{\phi(\tau_i^{q_1-1}, \sigma_k^{l+1})}{\tau_i^{q_1-1} \sigma_k^{l+1}} \right] Q_i^{q_1-1} + \\ &+ P_k^{q_2-1} \sum_{j=1}^{q_1-2} \left[\frac{\phi(\tau_i^j, \sigma_k^{q_2-1})}{\tau_i^j \sigma_k^{q_2-1}} - \frac{\phi(\tau_i^{j+1}, \sigma_k^{q_2-1})}{\tau_i^{j+1} \sigma_k^{q_2-1}} \right] Q_i^j + P_k^{q_2-1} Q_i^{q_1-1} \frac{\phi(\tau_i^{q_1-1}, \sigma_k^{q_2-1})}{\tau_i^{q_1-1} \sigma_k^{q_2-1}} =: \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{19}$$

where $\quad P_k^l = \sum_{r=1}^{l} \sin m\sigma_k^r.$

We start with J_1

$$\begin{aligned} J_1 &= \sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\left[\frac{\phi(\tau_i^j,\sigma_k^l)}{\tau_i^j\sigma_k^l}-\frac{\phi(\tau_i^j,\sigma_k^{l+1})}{\tau_i^j\sigma_k^{l+1}}-\frac{\phi(\tau_i^{j+1},\sigma_k^l)}{\tau_i^{j+1}\sigma_k^l}+\frac{\phi(\tau_i^{j+1},\sigma_k^{l+1})}{\tau_i^{j+1}\sigma_k^{l+1}}\right]Q_i^jP_k^l = \\ &= \sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\left[\frac{\Delta_{12}\phi(\tau_i^j,\sigma_k^l)}{\tau_i^j\sigma_k^l}+\frac{\Delta_1\phi(\tau_i^j,\sigma_k^{l+1})p_2h_M}{\tau_i^j\sigma_k^l\sigma_k^{l+1}}+\frac{\Delta_2\phi(\tau_i^{j+1},\sigma_k^l)p_1h_N}{\tau_i^j\tau_i^{j+1}\sigma_k^l}+\right. \\ &+ \left.\frac{p_1h_Np_2h_M\phi(\tau_i^{j+1},\sigma_k^{l+1})}{\tau_i^j\tau_i^{j+1}\sigma_k^l\sigma_k^{l+1}}\right]Q_i^jP_k^l =: K_1+K_2+K_3+K_4, \end{aligned} \tag{20}$$

where

$\Delta_1\phi(\tau_i^j,\sigma_k^{l+1})=\phi(\tau_i^{j+1},\sigma_k^{l+1})-\phi(\tau_i^j,\sigma_k^{l+1})$,

$\Delta_2\phi(\tau_i^{j+1},\sigma_k^l)=\phi(\tau_i^{j+1},\sigma_k^{l+1})-\phi(\tau_i^{j+1},\sigma_k^l)$,

$\Delta_{12}\phi(\tau_i^j,\sigma_k^l)=\Delta_1\Delta_2\phi(\tau_i^j,\sigma_k^l)=\phi(\tau_i^{j+1},\sigma_k^{l+1})-\phi(\tau_i^j,\sigma_k^{l+1})-\phi(\tau_i^{j+1},\sigma_k^l)+\phi(\tau_i^j,\sigma_k^l)$.

Using inequalities $\tau_i^j\ge jp_1h_N,\ \sigma_k^l\ge lp_2h_M,\ |Q_i^j|<2,\ |P_k^l|<2$, we get

$$|K_1|\le\frac{4}{p_1h_Np_2h_M}\sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\frac{|\Delta_{12}\phi(\tau_i^j,\sigma_k^l)|}{jl}\le\frac{C}{p_1h_Np_2h_M}U_{n,m}(F). \tag{21}$$

Similarly, for K_2 we obtain

$$\begin{aligned} |K_2| &\le \sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\frac{4}{l(l+1)p_2h_Mp_1h_N}\cdot\frac{|\Delta_1\phi(\tau_i^j,\sigma_k^{l+1})|}{j}= \\ &= \frac{4}{p_2h_Mp_1h_N}\sum_{l=1}^{q_2-2}\frac{1}{l(l+1)}\sum_{j=1}^{q_1-2}\frac{|\Delta_1\phi(\tau_i^j,\sigma_k^{l+1})|}{j}\le\frac{C}{p_1h_Np_2h_M}U_{n,m}(F). \end{aligned} \tag{22}$$

Absolutely same proof applies for K_3. As for K_4 we have

$$|K_4| \le \frac{4}{p_1h_Np_2h_M}\sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\frac{|\phi(\tau_i^{j+1},\sigma_k^{l+1})|}{j(j+1)l(l+1)}. \tag{23}$$

This yields

$$\begin{aligned} p_1p_2h_Nh_M|K_4| &\le \sum_{l=1}^{q_2-2}\sum_{j=1}^{q_1-2}\frac{|\phi(\tau_i^{j+1},\sigma_k^{l+1})|}{j^2l^2}=\sum_{l=1}^{[\ln m]}\sum_{j=1}^{[\ln n]}\frac{|\phi(\tau_i^{j+1},\sigma_k^{l+1})|}{j^2l^2}+ \\ &+ \sum_{l=1}^{[\ln m]}\sum_{j=[\ln n]+1}^{\infty}\frac{|\phi(\tau_i^{j+1},\sigma_k^{l+1})|}{j^2l^2}+\sum_{l=[\ln m]+1}^{\infty}\sum_{j=1}^{\infty}\frac{|\phi(\tau_i^{j+1},\sigma_k^{l+1})|}{j^2l^2}= \\ &= C\left[\omega\left(F,\frac{\ln n}{n}+\frac{\ln m}{m}\right)+\|F\|_\infty(\eta_n+\eta_m)\right]\le C\cdot U_{n,m}(F), \end{aligned} \tag{24}$$

where $\eta_n,\eta_m\to 0$ when $n,m\to\infty$. From (20)–(24) we get

$$p_1p_2h_Nh_M|J_1|=O\left(U_{n,m}(F)\right). \tag{25}$$

We proceed with the estimation of J_2:

$$\begin{aligned} |J_2| &\le \frac{|Q_i^{q_1-1}|}{\tau_i^{q_1-1}}\sum_{l=0}^{q_2-2}\left|\frac{\phi(\tau_i^{q_1-1},\sigma_k^l)}{\sigma_k^l}-\frac{\phi(\tau_i^{q_1-1},\sigma_k^{l+1})}{\sigma_k^{l+1}}\right||P_k^l|\le \\ &\le \frac{|Q_i^{q_1-1}|}{\tau_i^{q_1-1}}\sum_{l=0}^{q_2-2}\left|\frac{\phi(\tau_i^{q_1-1},\sigma_k^l)-\phi(\tau_i^{q_1-1},\sigma_k^{l+1})}{\sigma_k^l}\right||P_k^l|+ \end{aligned}$$

$$+ \quad \frac{|Q_i^{q_1-1}|}{\tau_i^{q_1-1}} \sum_{l=0}^{q_2-2} |\phi(\tau_i^{q_1-1}, \sigma_k^{l+1})| \left[\frac{1}{\sigma_k^l} - \frac{1}{\sigma_k^{l+1}}\right] |P_k^l| \le$$

$$\le \quad \frac{C}{(q_1-1)p_1 h_N p_2 h_M} \sum_{l=0}^{q_2-2} \left| \frac{\phi(\tau_i^{q_1-1}, \sigma_k^l) - \phi(\tau_i^{q_1-1}, \sigma_k^{l+1})}{l} \right| +$$

$$+ \quad \frac{C}{(q_1-1)p_1 h_N} \|\phi\|_{C((0,\pi)^2)} \sum_{l=0}^{q_2-2} \left[\frac{1}{\sigma_k^l} - \frac{1}{\sigma_k^{l+1}}\right] \le$$

$$\le \quad \frac{C}{(q_1-1)p_1 h_N p_2 h_M} \left[O(W_{n,m}(\phi,[0,\pi]^2)) + \|\phi\|_{C((0,\pi)^2)}\right],$$

which together with $q_1 \to \infty, p_1 \to \infty$ implies

$$p_1 p_2 h_N h_M |J_2| = O(U_{n,m}(F)). \tag{26}$$

Estimation of J_3 is done in the same way. For J_4 we have

$$|J_4| = \left| P_k^{q_2-1} Q_i^{q_1-1} \frac{\phi(\tau_i^{q_1-1}, \sigma_k^{q_2-1})}{\tau_i^{q_1-1} \sigma_k^{q_2-1}} \right| \le \frac{4}{(q_1-1)(q_2-1)p_1 h_N p_2 h_M} \|\phi\|_{C((0,\pi^2)},$$

which together with $q_1, q_2 \to \infty$ when $n,m \to \infty$ gives us

$$p_1 p_2 h_N h_M |J_4| = O(U_{n,m}(F)). \tag{27}$$

Now, from (19) and (25)–(27) we get

$$p_1 p_2 h_N h_M |J| = O(U_{n,m}(F)), \tag{28}$$

which combined with (13)–(18) completes the proof of Lemma . □

The next lemma was proved in [3].

L e m m a 3 . 3 [3]. For any function $\omega(\delta)$, which satisfies the conditions (1), there exists a homeomorphism $\tau(t)$ of the interval T such that for all F,

$$F(x,y) = f(\tau(x), \tau(y)), \quad f \in C(T^2), \quad \omega(\delta, f) \le \omega(\delta)$$

the following conditions hold:

$$V(F) < \infty, \quad \lim_{h \to 0} V_h(F) = 0. \tag{29}$$

It is easy to see that Lemmas 3.1 and 3.2 imply Theorem 1.6, while from Lemmas 3.1, 3.2 and 3.3 follows Theorem 1.7, since (29) implies

$$\lim_{n,m \to \infty} U_{n,m}(F) = 0.$$

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