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Mathematics

TOPOLOGIES ON THE GENERALIZED PLANE

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In this paper the topologies arising on the generalized plane Δ and its subsets are considered and their comparisons are investigated.

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1. Introduction. Let Γ be a subgroup of the group of real numbers $\mathbb{R}$, which is dense in $\mathbb{R}$ with respect to Euclidean topology τ and let G be the characters group of Γ: $G = \hat{\Gamma}$. By Pontryagin duality theorem we have that the characters group of G is isomorphic to Γ: $\hat{G} \cong \Gamma$. Using G we define a Cartesian product $G \times [0,\infty)$ and glue the bottom layer $G \times \{0\}$ to the point. The obtained space is called *generalized plane* and is denoted by Δ. Given construction is due to Arens and Singer. Let $\pi : G \times [0,\infty) \to \Delta$ be a canonical projection. Then the elements of Δ are the points $\pi(\alpha,r) = (\alpha,r)$ with $\alpha \in G$ and $r > 0$, and $* = \pi(G \times \{0\})$ is the null element of the space Δ. Generalized plane Δ can be also canonically identified with the space $\mathcal{C} = \{\alpha r : \alpha \in G, r \in [0,\infty)\}$, which is the analogue of the complex plane $\mathbb{C}$ consisting of the homorphisms $\alpha r : \Gamma \to \mathbb{C} : a \mapsto \alpha(a)r^a$. It is usually more convenient to take $\Delta = \mathcal{C}$, in which case the representation $s = \alpha r$ of an element $s \in \Delta$ is called a polar decomposition and the number r is called a modulus of s. As the null element $*$ essentially differs from the other elements of the space Δ, it makes sense to define the space $\Delta^0 = \Delta \setminus \{*\}$, so called punctured generalized plane.

Obviously, $\Delta^0 = G \times (0,\infty)$ and Δ^0 can be canonically identified with the space $\{\alpha r : \alpha \in G, r \in (0,\infty)\}$.

On the space Δ the theory of generalized analytic functions is developed, which allows to find new features by applying the classical apparatus of complex analysis (see [1–3]). Our goal is to investigate the topologies arising on Δ.

2. Topologies on Δ. Thus G is the group of characters of Γ. Let $\{T\}$ be some base for the open sets of the unit circle $\mathbb{T}$ of the complex plane $\mathbb{C}$ and let $\mathcal{F}$ be the

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collection of all finite subsets of Γ. Define $P(F,T) = \{\chi \in G |\ \chi(F) \subseteq T\}$. The family $\{P(F,T), F \in \mathcal{F}, T \in \{T\}\}$ is a base for some topology on G, which is called finite–open topology for obvious reasons and is denoted by k. Then the topology on Δ would be the standard quotient topology $\tau_\Delta = \{U \subset \Delta : \pi^{-1}(U) \in k \times \tau_{[0,\infty)}\}$, where $\tau_{[0,\infty)}$ is a restriction of the Euclidean topology τ to $[0,\infty)$. As a base for the topology τ_Δ could be taken the family of sets $\mathcal{B} = \{\pi(G \times [0,r))\}_{r>0} \cup \pi(\textit{any base for } G \times (0,\infty))$, where the first component in this union is the neighborhood base at the null element $*$ of the space Δ. Similarly, we define the topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$ on Δ^0. The canonical projection π is not open (since the topology k is not trivial), but it is a closed mapping that induces a homeomorphism

$$\pi\Big|_{G\times(0,\infty)} : (G \times (0,\infty), k \times \tau_{(0,+\infty)}) \to (\Delta^0, \tau_{\Delta^0}).$$

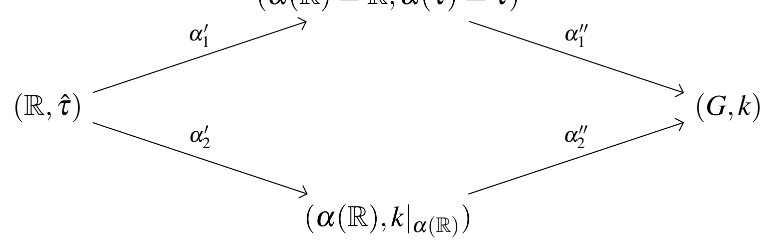
The space Δ is then a locally compact Hausdorff space.

Let us now consider the mapping $\alpha : \mathbb{R} \to G : t \to \alpha_t$, where $\alpha_t(a) = e^{iat}, a \in \Gamma$. The density of Γ in $\mathbb{R}$ implies that α is injective. Indeed, if $\alpha_{t_1} = \alpha_{t_2}$ with $t_1, t_2 \in \mathbb{R}, t_1 \neq t_2$, then $e^{iat_1} = e^{iat_2}$ for all $a \in \Gamma$. Since Γ is dense in $\mathbb{R}$ and $\alpha_{t_i}, i = 1,2$, are both continuous on $(\mathbb{R}, \tau)$, we get that $e^{iat_1} = e^{iat_2}$ for all $a \in \mathbb{R}$, and, therefore, $t_1 = t_2$. This argumentation can be also used as a justification of the parity $\alpha(\mathbb{R}) = \hat{\mathbb{R}}$ of two groups of characters with different domains (Γ and $\mathbb{R}$ respectively). In other words, the equality $\alpha(\mathbb{R}) = \hat{\mathbb{R}}$ matches an element $\alpha_t \in \alpha(\mathbb{R}), t \in \mathbb{R}$, with the element of $\hat{\mathbb{R}}$ corresponding to the number $t \in \mathbb{R}$. The proof of density of the image $\alpha(\mathbb{R})$ in G is similar and is based on the fact that $\alpha(\mathbb{R})$ separates the points of a group Γ [4].

The space $\Delta^0 = G \times (0,\infty)$, which has been canonically identified with the space $\{\alpha r : \alpha \in G, r \in (0,\infty)\}$, is a locally compact abelian group under the coordinate-wise multiplication with the unit element $\alpha_0 \cdot 1 = \alpha(0)$.

There are two topologies on $\alpha(\mathbb{R})$: the restriction $k|_{\alpha(\mathbb{R})}$ of the finite-open topology k on G and the topology $\hat{\tau}$, which arises as a compact-open topology on $\alpha(\mathbb{R}) = \hat{\mathbb{R}}$. Since each finite set is compact we get that the topology $\hat{\tau}$ is stronger than $k|_{\alpha(\mathbb{R})}$. As a neighborhood base at the unit element $\alpha_0 \in \alpha(\mathbb{R})$ (which determines $\hat{\tau}$) can be taken the family $\{P_\varepsilon\}_{\varepsilon\in(0,\pi)}$ of the sets $P_\varepsilon = \{\alpha_t : \alpha_t([-1,1]) \subset V_\varepsilon\}$, where $V_\varepsilon = \{\xi \in \mathbb{T} : \xi = e^{i\theta}, \theta \in (-\varepsilon, \varepsilon)\}$. Clearly $P_\varepsilon = \alpha((-\varepsilon, \varepsilon))$ and, therefore, $\hat{\tau} = \alpha(\tau)$ the homeomorphic image of the Euclidean topology τ on $\mathbb{R}$.

The mentioned topologies on $\alpha(\mathbb{R})$ determine two different factorizations of the mapping α, which are presented in the following diagram:



In this diagram α_1' is a homeomorphism, α_2' is a continuous homomorphism and the insertion $\alpha_1'' : \alpha_t \mapsto \alpha_t|\Gamma$ as well as the embedding α_2'' are continuous.

The group $\alpha(\mathbb{R})$ is a path-connected group in both topologies as the image of a path-connected space under the continuous mappings α_1' and α_2'. Moreover, we claim that these path-connectednesses are equivalent. Indeed, the path-connectedness of $\alpha(\mathbb{R})$ with respect to the topology $\hat{\tau}$ clearly implies the path-connectedness with respect to the weaker topology $k|_{\alpha(\mathbb{R})}$. Let us now prove the converse statement.

Lemma 2.1. Any path

$$\sigma : I = [0,1] \to \alpha(\mathbb{R}),$$

that is continuous with respect to the topology $k|_{\alpha(\mathbb{R})}$ is also continuous with respect to the topology $\hat{\tau} = \alpha(\tau)$.

Proof. Let us first prove the continuity of a path σ at the point $t_0 \in I$ with $\sigma(t_0) = \alpha_0$, using the fact that $\{P_\varepsilon\}_{\varepsilon\in(0,\pi)}$ is the neighborhood base at α_0 in a topology $\hat{\tau}$. Let us fix any $\varepsilon \in (0,\pi)$ and consider the corresponding

$$P_\varepsilon = \big\{\alpha_t : \alpha_t\big([-1,1]\big) \subset V_\varepsilon\big\} = \alpha\big((-\varepsilon,\varepsilon)\big)$$

as the neighborhood of α_0 in a topology $\hat{\tau}$. The set

$$Q_\varepsilon = \{\alpha_t : \alpha_t(1) \subset V_\varepsilon\} = \cup_{n\in\mathbb{Z}}\alpha(I_n)$$

is then a neighborhood of α_0 in a topology $k|_{\alpha(\mathbb{R})}$, where $I_n = (2\pi n - \varepsilon, 2\pi n + \varepsilon)$, $n \in \mathbb{Z}$. Obviously we have $P_\varepsilon \subset Q_\varepsilon$.

We have that σ is continuous at t_0 with respect to the topology $k|_{\alpha(\mathbb{R})}$. Therefore, there exists a $\delta > 0$ such that $\sigma(I_\delta) \subset Q_\varepsilon$, where $I_\delta = (t_0 - \delta, t_0 + \delta) \cap I$. We want to show that $\sigma(I_{\delta/2}) \subset P_\varepsilon$.

Indeed, the continuum $\sigma(\bar{I}_{\delta/2})$, which is contained in $\sigma(I_\delta)$, is covered by the set $\tilde{Q}_\varepsilon = \{\alpha_t : \alpha_t(1) \subset \overline{V}_\varepsilon\} = \cup_{n\in\mathbb{Z}}\alpha(\bar{I}_n)$, which is a countable union of compact and therefore closed sets $\alpha(\bar{I}_n)$. By Sierpinski's theorem (see [5]) at most one of these sets is non-empty. Since the set $\sigma(\bar{I}_{\delta/2})$ certainly intersects with $\alpha(\bar{I}_0)$, then $\alpha(\bar{I}_0)$ would be the mentioned unique non-empty set.

It follows that $\sigma(I_{\delta/2}) \subset \alpha(\bar{I}_0) \cap Q_\varepsilon = \alpha(I_0) = P_\varepsilon$, as desired.

The general case is reduced to the considered situation by the transitions from σ to $\tilde{\sigma}(t) = \sigma(t_0)^{-1}\sigma(t)$ and back again, using the fact that the shifts by $\sigma(t_0)^{-1}$ and $\sigma(t_0)$ are topological automorphisms of $\alpha(\mathbb{R})$ in both topologies $k\Big|_{\alpha(\mathbb{R})}$ and $\hat{\tau}$. □

The mapping $\alpha : \mathbb{R} \to G$ generates an embedding

$$\varphi : \mathbb{C} \to \Delta^0 : z = t + iy \mapsto \varphi_z = \alpha_t e^{-y}.$$

The transition from α to φ complexifies the above diagram keeping the properties of the mappings in it.

The topology $\tau_{\Delta^0}\Big|_{\varphi(\mathbb{C})}$, which is induced on $\varphi(\mathbb{C})$ by a topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$, is weaker than the topology $\tau_\varphi = \varphi(\tau_c)$, which is the homeomorphic image of the Euclidean topology τ_c on $\mathbb{C}$. The topology $\varphi(\tau_c)$ has two other equivalent descriptions: it emerges as a product of topologies $\hat{\tau} \times \tau_{(0,\infty)}$ with $\{P_\varepsilon \times (e^{-\delta}, e^{\delta})\}_{\varepsilon\in(0,\pi),\delta>0}$ being the neighborhood base at the unit element $(\alpha_0, 1) \cong \varphi(0)$ and as a compact–open topology on $\varphi(\mathbb{C})$ with the neighborhood base at the unit element formed by the sets

$$P_{\varepsilon,\delta} = \{\varphi_z : \varphi_z([-1,1]) \subset V_{\varepsilon,\delta}\} = \varphi(K_{\varepsilon,\delta}), \varepsilon \in (0,\pi),\ \ \delta > 0,$$

where $V_{\varepsilon,\delta} = \{w = \rho e^{i\theta} : e^{-\delta} < \rho < e^{\delta}, e^{i\theta} \in V_\varepsilon\}$, and the sets $K_{\varepsilon,\delta} = \{z = t + iy : |t| < \varepsilon, |y| < \delta\}$ obviously form the neighborhood base at the zero element $z = 0$ of a group $\mathbb{C}$.

Note that since $\alpha(\mathbb{R})$ is dense in G the image $\varphi(\mathbb{C})$ is dense in both Δ^0 and Δ.

Definition 2.1. For a point $s \in \Delta^0$ the set $\mathbb{C}_s = s\varphi(\mathbb{C})$ is called a plane in Δ^0 passing through s.

Obviously, $\mathbb{C}_s$ is dense in Δ^0 for any $s \in \Delta^0$. We also denote

$$\mathbb{C}_0 := \mathbb{C}_{\varphi(0)} = \varphi(\mathbb{C}).$$

Define a mapping $\varphi_s : \mathbb{C} \to \mathbb{C}_s : z \mapsto s\varphi_z$. Again there are two topologies on each plane $\mathbb{C}_s$: the topology $\tau_s := \tau_{\Delta^0}\Big|_{\mathbb{C}_s}$, which is induced from Δ^0 and the stronger topology $\tau_{s\varphi} = s\tau_\varphi = \{sU : U \in \tau_\varphi\}$, which is inherited from $\mathbb{C}$ by the mapping φ_s.

3. Topologies Induced by Coverings. The theory of Bohr–Riemann surfaces considers so called *thin* sets K in Δ [2] and investigates the finite-sheeted coverings ([6], §4) of the space $\Delta^* = \Delta^0 \setminus K$. So, we now pass to the situation which often arises in that theory.

Let $s \in \Delta^0$ and let K be a closed nowhere dense subset of Δ^0 such that the intersection $K \cap \mathbb{C}_s$ is a discrete set. Define $\Delta^* = \Delta^0 \setminus K$ and $\mathbb{C}_s^* = \mathbb{C}_s \cap \Delta^* = \mathbb{C}_s \setminus K$.

Let us consider the preimage $\pi^{-1}(\mathbb{C}_s^*)$ under unfolded, finite-sheeted covering $\pi : X \to \Delta^*$, where X is a topological space. There are two topologies that arise on $\pi^{-1}(\mathbb{C}_s^*)$: the topology $\tau_{s,X}$, which is induced from a topology τ_X of the space X and is locally homeomorphic to $\tau_s^* = \tau_s\Big|_{\mathbb{C}_s^*}$, and the topology $\tau_{s,\mathbb{C}}$, which base consists of the path-connected components of the sets from $\tau_{s,X}$. Thus, using the characterization of $\tau_{s\varphi}$ as a topology with a base consisted of the path-connected components of the sets from τ_s, we get that the restriction $\pi\Big|_{\pi^{-1}(\mathbb{C}_s^*)}$ induces two coverings of the punctured plane $\mathbb{C}_s^*$: $\pi_{s,X} : \tau_{s,X} \to \tau_s^*$ and $\pi_{s,\mathbb{C}} : \tau_{s,\mathbb{C}} \to \tau_{s\varphi}^*$.

Theorem 3.1. The path-connected components of a subspace $\pi^{-1}(\mathbb{C}_s^*)$ in the topology $\tau_{s,X}$ coincide with the path-connected components of a Riemann surface $\pi^{-1}(\mathbb{C}_s^*)$ in $\tau_{s,\mathbb{C}}$.

Proof. Let $x \in \pi^{-1}(\mathbb{C}_s^*)$ and let C_x and D_x be the path-connected components of the preimage $\pi^{-1}(\mathbb{C}_s^*)$ containing x in the topologies $\tau_{s,\mathbb{C}}$ and $\tau_{s,X}$ respectively. Since $\tau_{s,\mathbb{C}}$ is stronger than $\tau_{s,X}$ it follows that $C_x \subset D_x$. Let us proof the converse inclusion. Fix an arbitrary point $y \in D_x$ and connect it with x by a path

$$\gamma : I = [0,1] \to X,$$

which lies in D_x and which is continuous with respect to the topology $\tau_{s,X}$.

Then $\lambda = \pi \circ \gamma : I \to \mathbb{C}_s^*$ is a continuous path from $\tau\big|_I$ to τ_s^*. Temporarily forgetting about the stars we get that λ is a continuous path from $\tau\big|_I$ to τ_s and, therefore, $s^{-1}\lambda : I \to \mathbb{C}_0$ is a continuous path from $\tau\big|_I$ to $\tau_0 = \tau_{\Delta^0}\big|_{\mathbb{C}_0} \cong$ $\cong k\big|_{\alpha(\mathbb{R})} \times \tau_{(0,+\infty)}$.

Using the interpretation of a space Δ^0 as a Cartesian product $G \times (0,\infty)$ we get that the mapping

$$s^{-1}\lambda : I \to \mathbb{C}_0$$

is comprised of the pair of mappings $s^{-1}\lambda(t) = (\beta(t), r(t)), t \in I$, with $\beta : I \to \to \alpha(\mathbb{R})$ and $r : I \to (0, +\infty)$. Then, as we know (see, e.g., [5]), the mapping $s^{-1}\lambda$ is continuous if and only if β is a continuous mapping from $\tau\big|_I$ to $k\big|_{\alpha(\mathbb{R})}$ and r is a continuous mapping from $\tau\big|_I$ to $\tau_{(0,+\infty)}$. Thus, by Lemma 2.1, we get that the path $\beta : I \to \alpha(\mathbb{R})$ is continuous with respect to the topology $\hat{\tau} = \alpha(\tau)$ as well. This together with the arguments above shows that the mapping $s^{-1}\lambda(t) = (\beta(t),\ r(t))$, $t \in I$, is continuous with respect to the topology $\hat{\tau} \times \tau_{(0,+\infty)} \cong \tau_\varphi$, i.e. λ is a continuous path from $\tau\big|_I$ to $\tau_{s\varphi}$ and, therefore, to $\tau^*_{s\varphi}$ as well. The continuity of the path γ with respect to the topology $\tau_{s,\mathbb{C}}$ is obtained from the local homeomorphity of π as a covering $\pi_{s,\mathbb{C}} : \pi^{-1}(\mathbb{C}^*_s) \to \mathbb{C}^*_s$ from $\tau_{s,\mathbb{C}}$ to $\tau^*_{s\varphi}$. Thus, $y \in C_x$. This completes the proof of the Theorem. □

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REFERENCES

1. **Arens R., Singer I.M.** Generalized Analytic Functions. // Trans. Amer. Math. Soc., 1956, v. 81, № 2, p. 379–393.
2. **Grigoryan S.A.** Generalized Analytic Functions. // Uspekhi Mat. Nauk, 1994, v. 49, № 2, p. 3–43 (in Russian).
3. **Grigoryan S.A.** The Divisor of a Generalized Analytic Function. // Math. Notes, 1997, v. 61, № 5, p. 655–661 (in Russian).
4. **Morris S.A.** Pontryagin Duality and the Structure of Locally Compact Abelian Groups. Russian translation. M.: Mir, 1980.
5. **Engelking R.** General Topology. Russian translation. M.: Mir, 1986.
6. **Forster O.** Riemann Surfaces. Russian translation. M.: Mir, 1980.