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M a t h e m a t i c s

ON THE UNIQUENESS OF ALGEBRAIC CURVES

V. H. BAYRAMYAN *, H. A. HAKOPIAN **, S. Z. TOROYAN***

Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia

It is well-known that $N-1$ n-independent nodes uniquely determine curve of degree n, where $N = (1/2)(n+1)(n+2)$. We are interested in finding the minimal number of n-independent nodes determining uniquely curve of degree $k \le n-1$. In this paper we show that this number for $k = n-1$ is $N-4$.

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1. Introduction. Denote the space of all bivariate polynomials of total degree $\le n$ by Π_n:

$$\Pi_n = \left\{ \sum_{i+j\le n} a_{ij} x^i y^j \right\}.$$

We have that

$$N := N_n := \dim \Pi_n = \binom{n+2}{2}.$$

Consider a set of s distinct nodes

$$\mathcal{X}_s = \{(x_1,y_1),(x_2,y_2),\dots,(x_s,y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions

$$p(x_i,y_i) = c_i, \quad i = 1,\dots,s, \tag{1.1}$$

is called interpolation problem.

D e f i n i t i o n 1 . 1 . The interpolation problem with a set of nodes $\mathcal{X}_s$ and Π_n is called *n-poised*, if for any data $(c_1,\dots,c_s)$, there is a *unique* polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1).

The conditions (1.1) give a system of s linear equations with N unknowns (the coefficients of the polynomial p). The poisedness means that this system has a unique solution for arbitrary right side values. Therefore, a necessary condition of poisedness is $s = N$. If this condition holds, then we obtain from the linear system.

* E-mail: vahagnbayramyan@gmail.com, ** hakop@ysu.am, *** sofitoroyan@gmail.com

Proposition 1.1. A set of nodes $\mathcal{X}_N$ is n-poised, if and only if

$$p \in \Pi_n \text{ and } p\big|_{\mathcal{X}_N} = 0 \quad \Longrightarrow \quad p = 0.$$

A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$, if

$$p(x_i, y_i) = \delta_{ik},\ i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^\star = p_A^\star = p_{A,\mathcal{X}_s}^\star$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of $\mathcal{X}_s$ but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of n-independence (see [1,2]).

Definition 1.2. A set of nodes $\mathcal{X}$ is called *n-independent*, if all its nodes have n-fundamental polynomials. Otherwise, if a node has no n-fundamental polynomial, then $\mathcal{X}$ is called *n-dependent*.

Fundamental polynomials are linearly independent. Therefore, a necessary condition for n-independence of $\mathcal{X}_s$ is $s \le N$.

Suppose a node set $\mathcal{X}_s$ is n-independent. Then, by the Lagrange formula we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.2):

$$p = \sum_{i=1}^{s} c_i p_i^\star.$$

In view of this, we get readily that the node set $\mathcal{X}_s$ is n-independent, if and only if the interpolating problem (1.2) is *solvable*, meaning that for any data $(c_1, \ldots, c_s)$ there is a polynomial $p \in \Pi_n$ (not necessarily unique) satisfying the interpolation conditions (1.2). Also, in view of Proposition 1.1, we have that a set $\mathcal{X}_N$ is n-poised, if and only if it is n-independent.

Evidently, any subset of n-poised set is n-independent. According to the following lemma any n-independent set is a subset of some n-poised set (see e.g., [3], Lemma 2.1).

Lemma 1.1. Any n-independent set $\mathcal{X}_s$ with $s < N$ can be extended to an n-poised set.

Below a well-known construction of n-poised set is described (see [4,5]).

Definition 1.3. A set of $N = 1 + \cdots + (n+1)$ nodes is called Berzolari–Radon set for degree n, or briefly BR_n set, if there exist lines $l_1, l_2, \ldots, l_{n+1}$, such that the sets $l_1,\ l_2 \setminus l_1,\ l_3 \setminus (l_1 \cup l_2), \ldots,\ l_{n+1} \setminus (l_1 \cup \cdots \cup l_n)$ contain exactly $(n+1), n, n-1, \ldots, 1$ nodes respectively.

An algebraic curve in the plane is the zero set of some bivariate polynomial of degree at least 1. We use the same letter, say p, to denote the polynomial $p \in \Pi_n \setminus \Pi_0$ and the corresponding curve p defined by equation $p(x,y) = 0$.

According to the following well-known statement, there are no more than $n+1$ n-independent points in any line:

Proposition 1.2. Assume that l is a line and $\mathcal{X}_{n+1}$ is any subset of l containing $n+1$ points. Then we have that

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_{n+1}} = 0 \Rightarrow p = lr, \quad \text{where } r \in \Pi_{n-1}.$$

Denote

$$d := d(n,k) := \dim \Pi_n - \dim \Pi_{n-k} = \frac{k(2n+3-k)}{2}.$$

The following is a generalization of Proposition 1.2.

Proposition 1.3 ([6], Prop. 3.1). Let q be an algebraic curve of degree $k \le n$ without multiple components. Then the following hold:

i) any subset of q containing more than $d(n,k)$ nodes is n-dependent,

ii) any subset $\mathcal{X}_d$ of q containing exactly $d(n,k)$ nodes is n-independent, if and only if the following condition holds:

$$p \in \Pi_n \quad \text{and} \quad p\big|_{\mathcal{X}_d} = 0 \Rightarrow p = qr, \text{ where } r \in \Pi_{n-k}.$$

Suppose that $\mathcal{X}$ is an n-poised set of nodes and q is an algebraic curve of degree $k \le n$. Then of course any subset of $\mathcal{X}$ is n-independent too. Therefore, according to Proposition 1.3 i), at most $d(n,k)$ nodes of $\mathcal{X}$ can lie in the curve q. Let us mention that a special case of this when q is a set of k lines is proved in [7].

Next lemma follows readily from the fact that the Vandermonde determinant, i.e., the main determinant of the linear system described after Definition 1.1, is a continuous function of the nodes of $\mathcal{X}_N$ (see e.g., [8], Remark 1.14).

Lemma 1.2. Suppose $\mathcal{X}_N = \{(x_i, y_i)\}_{i=1}^N$ is n-poised. Then there is a positive number ε such that any set $\mathcal{X}'_N = \{(x'_i, y'_i)\}_{i=1}^N$, for which the distance between (x'_i, y'_i) and (x_i, y_i) is less than ε, is n-poised too.

Finally, let us bring a lemma that follows from a simple Linear Algebra argument (see e.g., [9], Lemma 2.10).

Lemma 1.3. Suppose that two different curves of degree k pass through all the nodes of $\mathcal{X}_s$. Then for any node $A \notin \mathcal{X}$ there is a curve of degree k passing through A and all the nodes of $\mathcal{X}$.

2. The Result. From Lemma 1.3 and Proposition 1.1 we get readily that any $N-1$ n-independent nodes uniquely determine a curve of degree n. Below we find the minimal number of n-independent nodes that uniquely determine the curve of degree $n-1$ passing through them.

Proposition 2.1. Assume that $\mathcal{X}$ is any set of $N-4$ n-independent nodes lying in a curve of degree $n-1$. Then the curve is determined uniquely. Moreover, there is a set $\mathcal{X}^*$ of $N-5$ n-independent nodes such that more than one curves of degree $n-1$ pass through all its nodes.

Proof. Let us start with the part “moreover”. Consider the part of Berzolari–Radon set BR_n belonging to $n-2$ lines there: $\ell_1, \ldots, \ell_{n-2}$, i.e.,

$$\mathcal{X}' = BR_n \cap [\ell_1 \cup \cdots \cup \ell_{n-2}].$$

We have that the set $\mathcal{X}'$ consists of $\#\mathcal{X}_0 = (n+1) + n + \cdots + 4 = N-6$ nodes. We get a desired set $\mathcal{X}^*$ by adding to these $N-6$ nodes an other node A from BR_n, i.e.,

$\mathcal{X}^* := \mathcal{X}' \cup \{A\}$. Now, we have that the set $\mathcal{X}^*$ is n-independent, since it is a subset of n-poised set BR_n, and $\#\mathcal{X}^* = N-5$. Finally, consider the curves of degree $n-1$ of the form ℓq_{n-2}, where ℓ is any line passing through A and $q_{n-2} = \ell_1 \cdots \ell_{n-2}$. It remains to notice that all these curves of degree $n-1$ pass through all the nodes of $\mathcal{X}^*$.

Now let us prove the first statement of the Proposition. Suppose by the way of contradiction, that there are two different curves of degree $n-1$ passing through all the nodes of $\mathcal{X}$. Let us extend the set $\mathcal{X}$, according to Lemma 1.1 till an n-poised set $\mathcal{X}_N$, by adding 4 nodes to $\mathcal{X}$ denoted by A and B_1, B_2, B_3. In view of Lemma 1.2, there is a curve $p_i \in \Pi_{n-1}$ passing through B_i and all the nodes of $\mathcal{X}$, $i = 1,2,3$.

Denote by ℓ_{ij} the line passing through the nodes B_i and B_j, $1 \le i \ne j \le 3$. Note that, in view of Lemma 1.2, we may assume that B_1, B_2, B_3 are chosen such that they are not collinear and the lines $\ell_{12}, \ell_{23}, \ell_{31}$ do not pass through any node of $\mathcal{X}$, i.e.,

$$\mathcal{X} \cap \ell_{ij} = \emptyset, \ 1 \le i < j \le 3. \tag{2.1}$$

Now we readily get the following three representations of the fundamental polynomial $p_A^\star$:

$$p_A^\star = \ell_{23} \cdot p_1,$$

$$p_A^\star = \ell_{31} \cdot p_2,$$

$$p_A^\star = \ell_{12} \cdot p_3.$$

Indeed, clearly the products in the right hand sides of the equalities are nonzero polynomials from Π_n that vanish at all the nodes of $\mathcal{X}_1 \setminus \{A\}$. On the other hand, in view of Proposition 1.1, they do not vanish at A since the set $\mathcal{X}_N$ is n-poised.

From the uniqueness of the fundamental polynomials of n-poised sets and the fact that the lines ℓ_{ij} are distinct, we obtain readily that

$$p_A^\star = \ell_{12}\ell_{23}\ell_{31} \cdot q, \text{ where } q \in \Pi_{n-3}.$$

From here we get, in view of (2.1), that q vanishes at all the nodes of $\mathcal{X}$. This is a contradiction, since according to Proposition 1.3 i), a polynomial from Π_{n-3} may vanish at most at $d(n, n-3) = N-10$ n-independent nodes. $\square$

At the end we would like to put forward

Conjecture 2.1. The minimal number of n-independent nodes determining uniquely curve of degree k, $k \le n-2$, equals $N - \binom{n-k+3}{2} + 2$.

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R E F E R E N C E S

1. **Eisenbud D., Green M., Harris J.** Cayley–Bacharach Theorems and Conjectures. // Bull. Amer. Math. Soc. (N.S.), 1996, v. 33, p. 295–324.
2. **Hakopian H., Malinyan A.** Characterization of n-Independent Sets with no More Than $3n$ Points. // Jaen J. Approx., 2012, v. 4, p. 121–136.

3. **Hakopian H., Jetter K., Zimmermann G.** Vandermonde Matrices for Intersection Points of Curves. // Jaen J. Approx., 2009, v. 1, p. 67–81.
4. **Berzolari L.** Sulla Determinazione di Una Curva o di Una Superficie Algebrica e su Alcune Questioni di Postulazione. // Rend. del R. Ist. Lombardo di Scienze e Lettere, 1914, v. 47, p. 556–564.
5. **Radon J.** Zur Mechanischen Kubatur. // Monatsh. Math., 1948, v. 52, p. 286–300.
6. **Rafayelyan L.** Poised Nodes Set Constructions on Algebraic Curves. // East J. on Approx., 2011, v. 17, p. 285–298.
7. **Carnicer J.M., Gasca M.** Planar Configurations with Simple Lagrange Interpolation Formulae. In: Mathematical Methods in Curves and Surfaces (eds. T. Lyche and L.L. Schumaker). Oslo: 2000, p. 1–8; Nashville: Vanderbilt University Press, 2001, p. 55–62.
8. **Hakopian H.** Multivariate Divided Differences and Multivariate Interpolation of Lagrange and Hermite Type. // J. Approx. Theory, 1982, v. 34, p. 286–305.
9. **Hakopian H., Jetter K., Zimmermann G.** The Gasca–Maeztu Conjecture for $n = 5$. // Numer. Math., 2014, v. 127, p. 685–713.