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Mathematics

INNER AUTOMORPHISMS OF NON-COMMUTATIVE ANALOGUES OF THE ADDITIVE GROUP OF RATIONAL NUMBERS

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It is proved that the inner automorphisms group of the group $A(m,n)$ are characteristic subgroup in $Aut(A(m,n))$ for all $m>1$ and odd $n \geq 1003$, where the groups $A(m,n)$ are known non-commutative analogues of the additive group of rational numbers.

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Introduction. The inner automorphism of an element g of a group G is denoted by i_g and is given by the formula $x^{i_g} = gxg^{-1}$ for all $x \in G$. The mapping $\pi : G \to Aut(G)$, taking each $g \in G$ to the inner automorphism i_g, is a homomorphism, $Ker(\pi) = Z(G)$ and $Im(\pi) = Inn(G)$. It is easy to check that the automorphism group $Aut(G)$ of a group G with trivial center also is a group with trivial center. Therefore, in this case, we obtain a sequence of embeddings

$$G \to Aut(G) \to Aut(Aut(G)) = Aut^2(G) \to Aut^3(G) \to \dots \tag{1}$$

This sequence is called the automorphism tower of group G. For a group with a complete automorphism group the length of the automorphism tower is 1. Recall, that a group is called *complete* if its center is trivial and each of its automorphism is inner. Examples of such groups are automorphism groups of absolutely free groups (see [1]), the automorphism groups of non-abelian free solvable group of finite rank [2], as well as automorphism groups of the free Burnside groups $B(m,n)$ of odd periods $n \geq 1003$ (see [3, 4]). According to the criterion of Burnside (see [5]), the group $Aut(G)$ of a group G with trivial center is complete, if and only if $Inn(G)$ is a characteristic subgroup of $Aut(G)$.

Consider the group

$$A(m,n) = \langle a_1, a_2, \dots a_m, d \,|\, a_j d = d a_j \;\; A^n = d \;\text{ for all }\; A^n \in \textstyle\bigcup_{i=1}^{\infty} \mathscr{E}_i \;\; 1 \leq j \leq m \rangle, \tag{2}$$

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where $n \geq 1003$ an arbitrary odd number, $m > 1$. These groups are constructed and studied in works [6, 7]. Obviously, the center of $A(m,n)$ isn't trivial. We will investigate the automorphism group of $A(m,n)$. As in [7] is proved, groups $A(m,n)$ are torsion-free, and any two non-trivial cyclic subgroups of $A(m,n)$ have a nontrivial intersection. The same holds for the additive group of rational numbers. The existence of a non-commutative group with mentioned property (Kontorovich problem), was proved in [6], after being an open question for a long time. Group $A(m,n)$ is also called non-commutative analogue of the group of rational numbers. Our main result is the following theorem.

Theorem. The inner automorphisms group of the groups $A(m,n)$ are characteristic subgroup in the group $Aut(A(m,n))$ of all automorphisms of $A(m,n)$ for all $m > 1$ and odd $n \geq 1003$.

The Proof of Main Result. Since the center of each group is a characteristic subgroup, then every automorphism α of group $A(m,n)$ induces an automorphism $\overline{\alpha}$ of group $B(m,n)$ in a natural way.

Lemma 1. The Homomorphism $f : Aut(A(m,n)) \to Aut(B(m,n))$ given by the formula $f(\alpha) = \overline{\alpha}$ is not surjective.

Proof. It is easy to check that the automorphism of group $B(m,n)$, which takes each free generator a_i to a_i^2, has no pre-image under the homomorphism f. □

Let Φ be an automorphism of $Aut(A(m,n))$ and $\Phi(Inn(A(m,n))) = H$. We are going to prove that $H = Inn(A(m,n))$.

Lemma 2. $Inn(A(m,n)) \cap H \neq \{1\}$.

Proof. Suppose that an automorphism α from $\Phi(Inn(A(m,n)))$ is commutating with each inner automorphism i_a, $a \in G$. It means that for every element $x \in G$ the equality $a^{-1}a^{\alpha}x^{\alpha}(a^{-1})^{\alpha}a = x^{\alpha}$ holds for all $a \in A(m,n)$. It turns out that element $a^{-1}a^{\alpha}$ belongs to the center of $A(m,n)$ for all $a \in A(m,n)$. Hence, we get $a^{\alpha} = az$ for some $z \in Z(A(m,n))$. But $\alpha^n = id$, therefor, $a = a^{\alpha^n} = azz^{\alpha}z^{\alpha^2}\cdots z^{\alpha^{n-1}}$ and, hence, we obtain $zz^{\alpha}z^{\alpha^2}\cdots z^{\alpha^{n-1}} = 1$. The center $Z(A(m,n))$ of $A(m,n)$ is an infinite cyclic group generated by element d. Then $d^{\alpha} = d$ or $d^{\alpha} = d^{-1}$, because the center is characteristic subgroup. If $d^{\alpha} = d$, then $z^{\alpha} = z$ and $zz^{\alpha}z^{\alpha^2}\cdots z^{\alpha^{n-1}} = z^n = 1$. This contradicts the condition that $A(m,n)$ is torsion-free. If $d^{\alpha} = d^{-1}$, then $z^{\alpha} = z^{-1}$ and we get $zz^{\alpha}z^{\alpha^2}\cdots z^{\alpha^{n-1}} = z = 1$ because n is odd . □

Thus, by Lemma 2, some elements of the normal subgroup H are inner automorphisms. We have $Inn(A(m,n)) \simeq B(m,n) \simeq Inn(B(m,n))$, because the groups $B(m,n)$ and $A(m,n)/Z(A(m,n))$ are isomorphic, see Eq. (2). Besides, if the subgroups $Inn(A(m,n))$ and $H \simeq B(m,n)$ are different, then none of them can be contained in the other as a subgroup by Corollary 1 [8]. Consider the set $P = \{a \in A(m,n) | i_a \in H\}$.

Lemma 3. (1) P is a normal subgroup in $A(m,n)$.

(2) The subgroup P is ϕ-invariant for each automorphism $\phi \in H$, that is $P^{\phi} = P$.

(3) The equality $a^{\phi^{n-1}} \cdots a^{\phi^2} \cdot a^{\phi} \cdot a = 1$ holds for any automorphism $\phi \in H$ and any element $a \in P$.

(4) The inclusion $a^{-1}a^{\phi} \in P$ holds for any element $a \in B(m,n)$ and for each

automorphism $\phi \in H$.

P r o o f. (1) From $i_a \in H$ it follows that $i_x \circ i_a \circ i_x^{-1} = i_{xax^{-1}} \in H$ for all $x \in B(m,n)$, because H is a normal subgroup of $AutB(m,n)$. Thus, the condition $a \in P$ implies that $xax^{-1} \in P$ for all $x \in B(m,n)$. This means that the subgroup P is normal in $B(m,n)$.

(2) From the normality of subgroup H it also follows that $\phi \circ i_a \circ \phi^{-1} = i_{a^\phi} \in H$ for arbitrary ϕ, $i_a \in H$. Hence, the inclusion $a \in P$ implies $a^\phi \in P$ for all $\phi \in H$. For the same reason we have $a^{\phi^{-1}} \in P$. Therefore, the equality $P^\phi = P$ holds for any $\phi \in H$, that is, the subgroup P is ϕ-invariant.

(3) By definition, we have $i_a \in H$ for all $a \in P$, and so the equality $(\phi^{-1} \circ i_a)^n = 1$ holds for all $\phi, i_a \in H$ because H is a group of exponent n. On the other hand,

$$(\phi^{-1} \circ i_a)^n = \phi^{-n} \circ i_{a^{\phi^{n-1}} \cdots a^{\phi^2} \cdot a^\phi \cdot a}.$$

Taking into account that $\phi^{-n} = 1$ we obtain the equality $i_{a^{\phi^{n-1}} \cdots a^{\phi^2} \cdot a^\phi \cdot a} = 1$. This is equivalent to the equality $a^{\phi^{n-1}} \cdots a^{\phi^2} \cdot a^\phi \cdot a = 1$ by virtue of triviality of the center of the group $B(m,n)$.

(4) The commutator $[i_a, \phi^{-1}]$ belongs in the intersection $Inn(B(m,n)) \cap H$ for all $a \in B(m,n)$ and $\phi \in H$, because both subgroups $Inn(B(m,n))$ and H are normal. On the other hand we have the identities

$$[i_a, \phi^{-1}] = i_{a^{-1}} \circ \phi \circ i_a \circ \phi^{-1} = i_{a^{-1}a^\phi}.$$

Thus, we get $i_{a^{-1}a^\phi} \in H$, which means that $a^{-1}a^\phi \in P$. □

Further, using Lemma 3 and repeating the arguments of Section 4 [3], we get that $H \cap Inn(A(m,n)) = Inn(A(m,n))$.

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