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Mathematics

ON A SOLUTIONS OF ONE CLASS OF ALMOST HYPOELLIPTIC EQUATIONS

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We prove, that if $P(D) = P(D_1, D_2) = \sum_\alpha \gamma_\alpha D_1^{\alpha_1} D_2^{\alpha_2}$ is an almost hypoelliptic regular operator, then for enough small $\delta > 0$ all the solutions of the equation $P(D)u = 0$ from $L_{2,\delta}(R^2)$ are entire analytical functions.

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Definitions and Statement of the Problem. We use standard notations: N is the set of natural numbers; $N_0 = N \cup \{0\}$; $N_0^n = N_0 \times \cdots \times N_0$ is the set of n-dimensional multi-indices; E^n and R^n are n-dimensional real coordinate spaces $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ respectively.

Let $\xi \in R^n$, $x \in E^n$ and $\alpha \in N_n^\alpha$. We put $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$ and $D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial \xi_j}$ or $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$. Denote $R_+^n = \{\xi \in R^n; \xi_j \geq 0, j = 1, \ldots, n\}$.

Let $B = \{\alpha^k\}$ be finite set of points from N_0^n. Any minimal convex polyhedron $\Re = \Re(B) \subset R_+^n$ including $B \cup \{0\}$ is called characteristic polyhedron or Newton polyhedron of the set B. Say a polyhedron $\Re$ is regular, if $\Re$ has vertex at the origin, vertices on each axes apart from the origin, and all outer (relative to $\Re$) normals of $(n-1)$-dimensional faces of $\Re$ have nonnegative coordinates. We say a polyhedron $\Re$ is completely regular, if all outer normals of such faces have only positive coordinates.

Let $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$ be linear differential operator with constant coefficients and $P(\xi) = \sum_\alpha \gamma_\alpha \xi^\alpha$ be the corresponding symbol (characteristic polynomial),

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where summation is performed over the following finite set of multi-indices $(P) = \{\alpha \in N_0^n, \gamma_\alpha \neq 0\}$.

The characteristic polyhedron $\Re = \Re(P)$ of a set (P) is said to be the characteristic polyhedron of the operator $P(D)$ $\big($polynomial $P(\xi)\big)$.

Definition 1. [1,2]. An operator $P(D)$ with a polyhedron $\Re = \Re(P)$ is called regular, if there exists a constant $C > 0$ such that

$$\sum_{\alpha \in \Re \cap N_0^n} |\xi^\alpha| \leq C(|P(\xi)| + 1), \ \forall \xi \in R^n.$$

Definition 2. [3]. An operator $P(D)$ $\big($polynomial $P(\xi)\big)$ is called almost hypoelliptic, if there exists a constant $C > 0$ such that for any $\beta \in N_0^n$

$$|D^\beta(\xi)| \leq C(|P(\xi)| + 1), \ \forall \xi \in R^n.$$

Any hypoelliptic operator is almost hypoelliptic [4]. The reverse assertion is not true. In [5] an example of a polynomial with a Newton regular polyhedron is shown, which is almost hypoelliptic but not hypoelliptic.

L. Hörmander solved the problem of infinitely differentiability of solutions of homogenous hypoelliptic equations [4]. He indicated also those classes of Jevrey, containing the solutions of the homogenous equation $P(D)u = 0$. Similar results for some class of non hypoelliptic operators were obtained by V. Burenkov [6]. In [7] multi-anisotropic classes of Jevrey were defined and proved that solutions of a homogenous hypoelliptic equation $P(D)u = 0$ with constant coefficients belong to such classes generated by operator $P(D)$.

For $\delta > 0$ we put $N(P,\delta) = \{u; u \in L_{2,\delta}(E^2), P(D)u = 0\}$, where

$$L_{2,\delta}(E^2) = \{f; fe^{-\delta|x|} \in L_2(E^2)\}.$$

Let us define also the following weighted Sobolev spaces

$$W^m_{2,\delta}(E^2) = \{u; |D^\alpha u| e^{-\delta|x|} \in L_2(E^2); \forall \alpha \in N_0^2, |\alpha| \leq m\},$$

where $m \in N_0$. $W^{\Re}_{2,\delta}(E^2) = \{u; |D^\alpha u| e^{-\delta|x|} \in L_2(E^2); \forall \alpha \in \Re \cap N_0^2\}$.

If operator $P(D)$ is almost hypoelliptic [5], then there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$

$$N(P,\delta) \subset \cap_{m=0}^\infty W^m_{2,\delta}(E^2).$$

Let $P_0(D) = P_0(D_1, D_2) = \sum_{\alpha \in N_0^2} \gamma_\alpha D_1^{\alpha_1} D_2^{\alpha_2}$ is a regular operator with the characteristic polyhedron

$$\Re(P_0) = \{\nu \in R_+^2, \nu_1 \leq m_1, \nu_2 \leq m_2\},$$

where $m_1, m_2 \in N_0$. Obviously, $\Re(P_0)$ is a regular polyhedron.

It was proved in [3] that arbitrary regular operator with a Newton regular polyhedron is almost hypoelliptic.

We denote by $A_0(E^2)$ the set of entire analytical functions of real variables (x_1, x_2).

Our goal is to show that for a sufficient small $\delta > 0$ we have $N(P_0,\delta) \subset A_0(E^2)$.

Preliminaries. We need to change the function $e^{-\delta|x|}$ by an equivalent smooth function for the proof of the main results. Paper [8] proves existence of weighted function $g_\delta(x) = g(\delta x)$, $\delta > 0$, satisfying the following conditions.

1) There exists a constant $c > 0$ such that

$c^{-1}e^{-\delta|x|} \leq g_\delta(x) \leq ce^{-\delta|x|}$, $x \in E^2$;

2) for any $\alpha \in N_0^2$ there exists a constant $c_\alpha > 0$ such that

$|D^\alpha g_\delta(x)| \leq c_\alpha \delta^{|\alpha|} g_\delta(x)$, $x \in E^2$;

3) let $T > 0$, $S_T = \{x; x \in E^2, |x| \leq T\}$ and $\sigma_1 = \sigma_1(\delta, T) = c^2 e^{\delta T}$, where c is the constant from 1). Then,

$\sup\limits_{y \in S_T} g_\delta(x+y) \leq \sigma_1 g_\delta(x)$, $x \in E^2$;

4) let $T > 0$ and $\sigma_2 = \sigma_2(\delta, T) = \sqrt{2}c^2 \max\{c_\alpha; |\alpha| = 1\}\delta T e^{\delta T}$. Then,

$\sup\limits_{y \in S_T} |g_\delta(x+y) - g_\delta(x)| \leq \sigma_2 g_\delta(x)$, $x \in E^2$,

where $c > 0$ is the constant from 1) and c_α is the constant from 2).

In [8] it is also proved that the weighted spaces

$\left\{f; fg_\delta \in L_2(E^2), D^\alpha f \cdot g_\delta \in L_2(E^2), \forall \alpha \in \mathfrak{R} \cap N_0^2\right\}$

are topologically equivalent to $W_{2,\delta}^{\mathfrak{R}}(E^2)$. More exactly.

Lemma 1 ([8], Lemma 2). Let $\mathfrak{R}$ be a regular polyhedron in N_0^2 and $\delta_0 > 0$. Then, for any $\delta \subset (0, \delta_0)$ there exist constants $C_1 = C_1(\delta_0) > 0$, $C_2 = C_2(\delta_0) > 0$ such that for all $\delta \in (0, \delta_0)$ the following estimates hold for $\forall u \in W_{2,\delta}^{\mathfrak{R}}(E^2)$:

$$C_1^{-1} \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|D^\alpha(ug_\delta)\|_{L_2} \leq \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|(D^\alpha u)g_\delta\|_{L_2} \leq C_1 \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|D^\alpha(ug_\delta)\|_{L_2}, \quad (1)$$

$$C_2^{-1} \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|D^\alpha(ug_\delta)\|_{L_2} \leq \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|(D^\alpha u)e^{-\delta|\alpha|}\|_{L_2} \leq C_2 \sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|D^\alpha(ug_\delta)\|_{L_2}. \quad (2)$$

Lemma 2. Let $P(D_1, D_2)$ be an almost hypoelliptic operator with the characteristic polynomial $\mathfrak{R}(P)$. There exists a constant $C > 0$ such that for a sufficiently small $\delta > 0$ the following holds

$$\sum_{\alpha \in \mathfrak{R} \cap N_0^2} \|(D^\alpha u)g_\delta\|_{L_2} \leq C\|ug_\delta\|_{L_2}, \ \forall u \in W_{2,\delta}^{\mathfrak{R}(P)}(E^2) \cap N(P, \delta).$$

P r o o f . First, note that $C_0^\infty(E^2)$ is dense in $W_{2,\delta}^{\Re(P)}(E^2)$ (see [5], Lemma 3). Then, in view of regularity of operator $P(D)$ and the Parseval equality, applying the Fourier transformation, we get for $u \in W_{2,\delta}^{\Re(P)}(E^2)$

$$\begin{aligned}
\sum_{\alpha\in\Re\cap N_0^2} \|(D^\alpha u)g_\delta\|_{L_2} &\leq C_1 \sum_{\alpha\in\Re\cap N_0^2} \|D^\alpha(ug_\delta)\|_{L_2} = \\
&= C_1 \sum_{\alpha\in\Re\cap N_0^2} \Big(\int_{E^2}\!\!\int |D^\alpha(ug_\delta(x))|^2 dx\Big)^{\frac{1}{2}} = \\
&= C_1 \sum_{\alpha\in\Re\cap N_0^2} \Big(\int_{R^2}\!\!\int |\xi^\alpha F(ug_\delta)(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \leq \\
&\leq C_2\Big(\int_{R^2}\!\!\int (|P(\xi)|^2+1)|F(ug_\delta)(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \leq \\
&\leq C_2\Big(\Big(\int_{R^2}\!\!\int (|P(\xi)|F(ug_\delta)(\xi)|^2 d\xi\Big)^{\frac{1}{2}} + \\
&+ \Big(\int_{R^2}\!\!\int (|F(ug_\delta)(\xi)|^2 d\xi)^{\frac{1}{2}}\Big) = \\
&= C_2\Big(\Big(\int_{E^2}\!\!\int (|P(D)(ug_\delta)(x)|^2 dx\Big)^{\frac{1}{2}} + \\
&+ \Big(\int_{E^2}\!\!\int (|(ug_\delta)(x))|^2 dx)^{\frac{1}{2}}\Big) = \\
&= C_2(\|P(D)(ug_\delta\|_{L_2} + \|ug_\delta\|_{L_2}\|),
\end{aligned}$$

where C_1 and C_2 are some constants and $F(u)$ is the Fourier transformation of the function u. We apply the Leibniz generalized formula and take into account that $(P^{(\alpha)}) \subset \Re(P)$ (since $\Re(P)$ is regular). Then, according to property 2) of g_δ with $\delta \in (0,1)$, we derive

$$\sum_{\alpha\in\Re\cap N_0^2} \Big\|(D^\alpha u)g_\delta\Big\|_{L_2} \leq C_2\Big(\Big\|(P(D)u)g_\delta\Big\|_{L_2} +$$

$$+\sum_{0\neq\alpha\in N_0^2} \Big\|\frac{P^{(\alpha)}(D)u(D^\alpha)g_\delta}{\alpha!}\Big\|_{L_2} + \Big\|ug_\delta\Big\|_{L_2}\Big) \leq C_3\Big(\sum_{\alpha\in\Re\cap N_0^2} \Big\|(D^\alpha u)g_\delta\Big\|_{L_2} + \Big\|ug_\delta\Big\|_{L_2}\Big),$$

where $C_3 > 0$ is some constant.

Thus, for $C_2\delta \leq \dfrac{1}{2}$ we obtain

$$\sum_{\alpha\in\Re\cap N_0^2} \|(D^\alpha u)g_\delta\|_{L_2} \leq 2C_3\|ug_\delta\|_{L_2},\ \forall u \in W_{2,\delta}^{\Re(P)}(R^2)\cap N(P,\delta). \qquad \square$$

L e m m a 3 . Let $P_0(D_1,D_2)$ be an almost hypoelliptic operator with the characteristic polyhedron $\Re(P_0)$ and a number $\delta_0 \in (0,1)$ is such that for all $\delta \in (0,\delta_0)$ the following inclusion $N(P_0,\delta) \subset \cap_{m=0}^\infty W_{2,\delta}^m(E^2)$ is true. Then, for all $\alpha \in j\Re(P_0)$, $j = 0,1,\ldots$, there exists a constant $C > 0$ such that

$$\|(D^\alpha u)g_\delta\|_{L_2} \leq (2C)^j\|ug_\delta\|_{L_2},\ \forall u \in N(P_0,\delta).$$

Proof. Let $\alpha \in j\Re(P_0)\cap N_0^2$. Then, there exists a $\beta \in \Re(P_0)\cap N_0$ and a $\gamma \in (\gamma-1)\Re(P_0)\cap N_0^2$ such that $\alpha=\beta+\gamma$. According to the conditions of Lemma, we have

$$D^{\gamma} \in W_{2,\delta}^{\Re(P)}(E^2)\cap N(P_0,\delta)$$

for $u \in W_{2,\delta}^{\Re(P_0)}(E^2)\cap N(P_0,\delta)$. Then, in view of Lemma 2, the following estimate holds

$$\|(D^{\alpha}u)g_{\delta}\|_{L_2} = \|D^{\beta}(D^{\gamma}u)g_{\delta}\|_{L_2} \le 2C\|(D^{\gamma}u)g_{\delta}\|_{L_2}.$$

By the similar argument, taking into account that $0\cdot\Re(P_0)=\{0\}$, after $(j-1)$ steps we get $\|(D^{\alpha}u)g_{\delta}\|_{L_2} \le (2C)^j\|ug_{\delta}\|_{L_2}$, $\forall u \in N(P_0,\delta)$. □

Corollary. Let the conditions of Lemma 3 be valid with some constant $C=C(u)>0$ for all $\alpha \in j\Re(P_0)\setminus(j-1)\Re(P_0)$, $j=1,2\ldots$ Then, the following holds

$$\|(D^{\alpha}u)g_{\delta}\|_{L_2} \le C^{|\alpha|+1},\ \forall u \in N(P_0,\delta).$$

Proof. The statement follows from Lemma 3 and the following observations that for all $\alpha \in j\Re(P_0)\setminus(j-1)\Re(P_0)$

$$(j-1)\min(m_1,m_2) \le \alpha_1+\alpha_2 \le j(m_1+m_2),\ j=1,2,\ldots,$$

and $\|ug_{\delta}\|_{L_2}<\infty$, $u \in N(P_0,\delta)$.

The Main Result.

Theorem. For any compact set $K\subset E^2$ and for any function $u \in N(P_0,\delta)$ the following estimate holds

$$\sup_{x\in K}|D^{\alpha}u(x)| \le C^{|\alpha|+1},\ \forall \alpha \in N_0^2,$$

where $C=C(K,u)$ is some constant and $\delta>0$ is sufficiently small.

Proof. There exists a constant $C>0$ that for all $u \in C^{\infty}(E^2)$ the following holds $\sup_{x\in K}|u(x)| \le \sum_{|\beta|\le 2}\int\int_{K_1}|D^{\beta}u|dx$, where $K_1 \supset\supset K$, $(\rho(\partial K_1,K)>0)$. Now, from Corollary we get $\sup_{x\in K}|(D^{\alpha}u)g_{\delta}| \le C_1^{|\alpha|+1}$, $\forall\alpha \in N_0^2$ with some constant $C_1>0$. Applying the property 1) of g_{δ} leads us to the following estimate

$$\sup_{x\in K}|D^{\alpha}u| \le C_2C^{|\alpha|+1} \le C_3^{|\alpha|+1},\ \forall\alpha \in N_0^2,$$

with some constants $C_2,C_3>0$. □

Remark. From this estimate we conclude that for a sufficiently small $\delta>0$ the following holds $N(P_0,\delta)\subset A_0$.

Applying similar observations with some modifications, we can prove for a sufficiently small $\delta>0$ and for any $f \in \Gamma_{\delta}^{a}(E^2)$, $a>1$, that

$$N(P_0,f,\delta) \equiv \{u, u\cdot e^{-\delta|x|} \in L_2(E^2), P(D)u=f\} \subset \Gamma_{\delta}^{a}(E^2),$$

where $\Gamma_{\delta}^{a}(E^2)=\left\{f;\|D^{\alpha}f\cdot e^{-\delta|\alpha|}\|_{L_2} \le C^{|\alpha|+1}|\alpha|^{a|\alpha|}\right\}$.

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