

MOORE–PENROSE INVERSE OF BIDIAGONAL MATRICES. I

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In the present paper we deduce closed form expressions for the entries of the Moore–Penrose inverse of a special type upper bidiagonal matrices. On the base of the formulae obtained, a finite algorithm with optimal order of computational complexity is constructed.

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Introduction. For a real $m \times n$ matrix A , the Moore–Penrose inverse A^+ is the unique $n \times m$ matrix that satisfies the following four properties:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (A^+A)^T = A^+A, \quad (AA^+)^T = AA^+$$

(see [1], for example). If A is a square nonsingular matrix, then $A^+ = A^{-1}$. Thus, the Moore–Penrose inversion generalizes ordinary matrix inversion. The idea of matrix generalized inverse was first introduced in 1920 by E. Moore [2], and was later rediscovered by R. Penrose [3]. The Moore–Penrose inverses have numerous applications in least-squares problems, regressive analysis, linear programming and etc. For more information on the generalized inverses see [1] and its extensive bibliography.

In this work we consider the problem of the Moore–Penrose inversion of square upper bidiagonal matrices

$$A = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix}. \quad (1)$$

A natural question arises: what caused an interest in pseudoinversion of such matrices? The reason is as follows. The most effective procedure of computing

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the Moore–Penrose inverse involves two main steps [4]. In the first step an original matrix by means of Householder transformations is reduced to an upper bidiagonal form. Thus, there arises the problem of pseudoinversion of bidiagonal matrices of the form (1). In the second step an iterative procedure, which is known as Golub–Reinsch algorithm is implemented [4]. The procedure, by means of Givens rotations generates a sequence of matrices which converge to a diagonal matrix. As a result, at a certain step of the process we get an approximation of the singular value decomposition of the original bidiagonal matrix, by means of which, using a well-known formula (see [1, 4], for example), the Moore–Penrose inverse of this matrix is found.

In this paper we develop an approach, which allows to deduce formulae for the entries of the Moore–Penrose inverse of upper bidiagonal matrices. Obtained closed form solution for the Moore–Penrose inversion may be considered as alternative to sufficiently labour-consuming Golub–Reinsch iterative procedure. Moreover, explicit expressions for the entries of the Moore–Penrose inverse lead to a simple algorithm of their computation.

From now on, we will assume that

$$b_1, b_2, \dots, b_{n-1} \neq 0. \quad (2)$$

Otherwise, if some of over-diagonal entries of the matrix A from (1) are equal to zero, then the original problem is decomposed into several similar problems for bidiagonal matrices of lower order. Next, we assume that the bidiagonal matrix (1) is singular, i.e. $d_1 d_2 \dots d_n = 0$. We begin by considering the special case, where

$$d_1, d_2, \dots, d_{n-1} \neq 0, d_n = 0. \quad (3)$$

This particular case forms the problem solution basis in general case, that is, for an arbitrary arrangement of one or more zeros on the main diagonal of the matrix (1).

The Idea of the Derivation of Pseudoinversion Formulae. An approach to derive the formulae is based upon the well-known equality

$$A^+ = \lim_{\varepsilon \rightarrow +0} (A^T A + \varepsilon I)^{-1} A^T, \quad (4)$$

where I is an identity matrix, which holds true for any real $m \times n$ matrix [1]. Proposed way to obtain the matrix A^+ for the matrix (1) consists of the following. Finding first the inverse matrix $(A^T A + \varepsilon I)^{-1}$, the entries of the matrix $(A^T A + \varepsilon I)^{-1} A^T$ are computed and the character of their dependence on the parameter ε is revealed. Then, according to the equality (4), passing to the limit when $\varepsilon \rightarrow +0$, we arrive to a closed form expressions for the entries of the matrix A^+ . Mention that a similar approach was also used in the papers [5, 6].

Thus, the first problem that arise is the inversion of the matrix $A^T A + \varepsilon I$. For our bidiagonal matrix (1), under the assumption $d_n = 0$ (see (3)), we have

$$A^T A + \varepsilon I = \begin{bmatrix} d_1^2 + \varepsilon & b_1 d_1 & & & & & & \\ b_1 d_1 & b_1^2 + d_2^2 + \varepsilon & b_2 d_2 & & & & & 0 \\ & & \ddots & \ddots & & & & \\ 0 & & & & b_{n-2} d_{n-2} & & & \\ & & & & & b_{n-2}^2 + d_{n-1}^2 + \varepsilon & b_{n-1} d_{n-1} & \\ & & & & & b_{n-1} d_{n-1} & b_{n-1}^2 + \varepsilon & \end{bmatrix}. \quad (5)$$

To invert this matrix, let us take advantage of an algorithm developed in [7].

Consider a nonsingular symmetric tridiagonal matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & & & & \\ c_{21} & c_{22} & c_{23} & & & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & & c_{n-1n-2} & c_{n-1n-1} & c_{n-1n} & \\ & & & c_{nn-1} & c_{nn} & \end{bmatrix}, \quad (6)$$

where $c_{ii-1} = c_{i-1i} \neq 0, i = 2, 3, \dots, n$. Referring to [7], the matrix $C^{-1} = [x_{ij}]_{n \times n}$ can be obtained by the following computational procedure.

Algorithm 3d/inv ($C \Rightarrow C^{-1}$)

1. Compute the quantities $f_i (i = 2, 3, \dots, n)$, $g_i (i = 2, 3, \dots, n - 1)$ and $h_i (i = 1, 2, \dots, n - 1)$:

$$f_i = \frac{c_{ii}}{c_{ii-1}}, \quad g_i = \frac{c_{ii+1}}{c_{ii-1}}, \quad h_i = \frac{c_{ii}}{c_{ii+1}}. \quad (7)$$

2. Compute recursively the quantities:

$$\mu_n = 1, \quad \mu_{n-1} = -f_n, \quad (8)$$

$$\mu_i = -f_{i+1}\mu_{i+1} - g_{i+1}\mu_{i+2}, \quad i = n - 2, n - 3, \dots, 1.$$

3. Compute recursively the quantities:

$$v_1 = 1, \quad v_2 = -h_1, \quad (9)$$

$$v_i = -h_{i-1}v_{i-1} - \frac{1}{g_{i-1}}v_{i-2}, \quad i = 3, 4, \dots, n.$$

4. Compute the quantity $t = (c_{11}\mu_1 + c_{12}\mu_2)^{-1}$. (10)

Note: if C is nonsingular matrix then $c_{11}\mu_1 + c_{12}\mu_2 \neq 0$ [7].

5. The entries of the lower triangular part of the matrix C^{-1} are computed by:

$$x_{ij} = \mu_i v_j t, \quad i = j, j + 1, \dots, n; \quad j = 1, 2, \dots, n. \quad (11)$$

6. The entries of the upper triangular part of the matrix C^{-1} are computed by:

$$x_{ij} = \mu_j v_i t, \quad i = 1, 2, \dots, j - 1; \quad j = 2, 3, \dots, n. \quad (12)$$

End.

Consider as the matrix C our tridiagonal matrix $A^T A + \varepsilon I$. Comparing the records of matrices (5) and (6), we have

$$c_{ii} = b_{i-1}^2 + d_i^2 + \varepsilon, \quad i = 1, 2, \dots, n \quad (13)$$

(for the purpose of unification the record of the formulae, we set $b_0 = 0$) and

$$c_{ii+1} = b_i d_i, \quad i = 1, 2, \dots, n - 1; \quad c_{ii-1} = b_{i-1} d_{i-1}, \quad i = 2, 3, \dots, n. \quad (14)$$

The Entries of the Inverse Matrix $(A^T A + \varepsilon I)^{-1}$. Let us carry out a more detailed study of the quantities successively computed in the algorithm **3d/inv**. More precisely, we are interested in revealing the nature of dependence of these quantities on the parameter ε .

Consider first the quantities f_i, g_i, h_i , which were introduced in (7). Using the expressions (13) and (14), we get

$$f_i = \overset{\circ}{f}_i + \alpha_i \varepsilon, \quad i = 2, 3, \dots, n, \quad \text{where} \quad \overset{\circ}{f}_i = \frac{b_{i-1}^2 + d_i^2}{b_{i-1}d_{i-1}}, \quad \alpha_i = \frac{1}{b_{i-1}d_{i-1}}; \quad (15)$$

$$g_i = \frac{b_i d_i}{b_{i-1}d_{i-1}}, \quad i = 2, 3, \dots, n-1; \quad (16)$$

$$h_i = \overset{\circ}{h}_i + \beta_i \varepsilon, \quad i = 1, 2, \dots, n-1, \quad \text{where} \quad \overset{\circ}{h}_i = \frac{b_{i-1}^2 + d_i^2}{b_i d_i}, \quad \beta_i = \frac{1}{b_i d_i}. \quad (17)$$

Next, go to the quantities μ_i and v_i recursively defined in (8) and (9) respectively.

L e m m a 1. The quantities μ_i may be represented as

$$\mu_i = \overset{\circ}{\mu}_i + \gamma_i \varepsilon + O(\varepsilon^2), \quad i = 1, 2, \dots, n, \quad (18)$$

where the quantities $\overset{\circ}{\mu}_i$ and γ_i satisfy the following recurrence relations:

$$\begin{aligned} \overset{\circ}{\mu}_n &= 1, \quad \overset{\circ}{\mu}_{n-1} = -\overset{\circ}{f}_n, \\ \overset{\circ}{\mu}_i &= -\overset{\circ}{f}_{i+1} \overset{\circ}{\mu}_{i+1} - g_{i+1} \overset{\circ}{\mu}_{i+2}, \quad i = n-2, n-3, \dots, 1, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \gamma_n &= 0, \quad \gamma_{n-1} = -\alpha_n, \\ \gamma_i &= -\overset{\circ}{f}_{i+1} \gamma_{i+1} - g_{i+1} \gamma_{i+2} - \alpha_{i+1} \overset{\circ}{\mu}_{i+1}, \quad i = n-2, n-3, \dots, 1. \end{aligned} \quad (20)$$

P r o o f. Since $\mu_n = 1$, then in (18) for $i = n$ we set $\overset{\circ}{\mu}_n = 1, \gamma_n = 0$. Further, $\mu_{n-1} = -f_n$ (see (8)). According to the expressions (15), we have $f_n = \overset{\circ}{f}_n + \alpha_n \varepsilon$. Therefore, in the representation (18) for $i = n-1$ we set $\overset{\circ}{\mu}_{n-1} = -\overset{\circ}{f}_n, \gamma_{n-1} = -\alpha_n$.

Required representations for the indices in the range $1 \leq i \leq n-2$ can be readily derived by induction from the relations (8), using expressions (15). Indeed, having

$$\begin{aligned} \mu_i &= -f_{i+1} \mu_{i+1} - g_{i+1} \mu_{i+2} = \\ &= -(\overset{\circ}{f}_{i+1} + \alpha_{i+1} \varepsilon)(\overset{\circ}{\mu}_{i+1} + \gamma_{i+1} \varepsilon + O(\varepsilon^2)) - g_{i+1}(\overset{\circ}{\mu}_{i+2} + \gamma_{i+2} \varepsilon + O(\varepsilon^2)) = \\ &= (-\overset{\circ}{f}_{i+1} \overset{\circ}{\mu}_{i+1} - g_{i+1} \overset{\circ}{\mu}_{i+2}) + (-\overset{\circ}{f}_{i+1} \gamma_{i+1} - g_{i+1} \gamma_{i+2} - \alpha_{i+1} \overset{\circ}{\mu}_{i+1}) \varepsilon + O(\varepsilon^2), \end{aligned}$$

we get (18), as well as the recurrence relations (19) and (20). \square

At the same time, the quantities $\overset{\circ}{\mu}_i$ computed by the recursion (19) may be represented in closed form.

Let us introduce the following notation:

$$r_s \equiv \frac{b_s}{d_s}, \quad s = 1, 2, \dots, n-1. \quad (21)$$

Additionally, we set $r_0 = r_n = 1$.

L e m m a 2. The quantities $\overset{\circ}{\mu}_i$ may be written as

$$\overset{\circ}{\mu}_i = (-1)^{n-i} \prod_{s=i}^{n-1} r_s, \quad i = 1, 2, \dots, n. \quad (22)$$

Proof. Firstly, the value $\overset{\circ}{\mu}_n = 1$ conforms to the record (22). Then, in accordance with (15),

$$\overset{\circ}{\mu}_{n-1} = -\overset{\circ}{f}_n = -\frac{b_{n-1}}{d_{n-1}} = -r_{n-1}.$$

Further reasonings are carried out by induction. Taking into account the expressions (15) and (16), we have

$$\begin{aligned} \overset{\circ}{\mu}_i &= -\frac{b_i^2 + d_{i+1}^2}{b_i d_i} (-1)^{n-i-1} \prod_{s=i+1}^{n-1} r_s - \frac{b_{i+1} d_{i+1}}{b_i d_i} (-1)^{n-i-2} \prod_{s=i+2}^{n-1} r_s \\ &= (-1)^{n-i} \prod_{s=i+2}^{n-1} r_s \left(\frac{b_i^2 + d_{i+1}^2}{b_i d_i} r_{i+1} - \frac{b_{i+1} d_{i+1}}{b_i d_i} \right) = (-1)^{n-i} \prod_{s=i}^{n-1} r_s, \end{aligned}$$

which completes the proof of the Lemma. \square

The next assertion is a simple consequence of the equality (22).

Corollary 1. The following relation holds

$$\overset{\circ}{\mu}_i = -r_i \overset{\circ}{\mu}_{i+1}, \quad i = 1, 2, \dots, n-1. \quad (23)$$

A representation similar to (18) takes place also for the quantities v_i . It can be obtained using the relations (9) and the expressions (17).

Lemma 3. The quantities v_i may be represented as

$$v_i = \overset{\circ}{v}_i + \delta_i \varepsilon + O(\varepsilon^2), \quad i = 1, 2, \dots, n, \quad (24)$$

where the quantities $\overset{\circ}{v}_i$ and δ_i satisfy the following recurrence relations:

$$\begin{aligned} \overset{\circ}{v}_1 &= 1, \quad \overset{\circ}{v}_2 = -\overset{\circ}{h}_1, \\ \overset{\circ}{v}_i &= -\overset{\circ}{h}_{i-1} \overset{\circ}{v}_{i-1} - \frac{1}{g_{i-1}} \overset{\circ}{v}_{i-2}, \quad i = 3, 4, \dots, n, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \delta_1 &= 0, \quad \delta_2 = -\beta_1, \\ \delta_i &= -\overset{\circ}{h}_{i-1} \delta_{i-1} - \frac{1}{g_{i-1}} \delta_{i-2} - \beta_{i-1} \overset{\circ}{v}_{i-1}, \quad i = 3, 4, \dots, n. \end{aligned} \quad (26)$$

The quantities $\overset{\circ}{v}_i$ may be represented in closed form as well.

Lemma 4. The quantities $\overset{\circ}{v}_i$ may be written by

$$\overset{\circ}{v}_i = (-1)^{i+1} \prod_{s=1}^{i-1} \frac{1}{r_s}, \quad i = 1, 2, \dots, n. \quad (27)$$

The proof of the Lemma is similar to the one of Lemma 2, using relations (25) and expressions (16), (17). As a simple consequence of the equality (27) we obtain the following statement.

Corollary 2. The following relation holds

$$\overset{\circ}{v}_{i+1} = -\frac{1}{r_i} \overset{\circ}{v}_i, \quad i = 1, 2, \dots, n-1. \quad (28)$$

Our next task is to get the expression for the quantity t from (10), depending on the parameter ε . Since $c_{11} = d_1^2 + \varepsilon$, $c_{12} = b_1 d_1$ (compare (5) and (6)), then taking into account the representations (18) for the quantities μ_i , we get

$$\begin{aligned} t &= ((d_1^2 + \varepsilon)(\overset{\circ}{\mu}_1 + \gamma_1 \varepsilon + O(\varepsilon^2)) + b_1 d_1(\overset{\circ}{\mu}_2 + \gamma_2 \varepsilon + O(\varepsilon^2)))^{-1} = \\ &= (d_1^2(\overset{\circ}{\mu}_1 + r_1 \overset{\circ}{\mu}_2) + (\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1))\varepsilon + O(\varepsilon^2))^{-1}. \end{aligned}$$

By virtue of the relation (23), $\overset{\circ}{\mu}_1 + r_1 \overset{\circ}{\mu}_2 = 0$. Therefore,

$$t = ((\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1))\varepsilon + O(\varepsilon^2))^{-1}. \quad (29)$$

Finally, having the representations for the quantities μ_i , v_i and t , we can obtain the entries of the inverse matrix $(A^T A + \varepsilon I)^{-1} = [x_{ij}]_{n \times n}$ and reveal their dependence on the parameter ε . It may be done by the equalities (11) and (12), given in the algorithm **3d/inv**. We leave it to the next section.

Formulae for the Entries of the Matrix A^+ . Let

$$(A^T A + \varepsilon I)^{-1} A^T \equiv Y(\varepsilon) = [y_{ij}(\varepsilon)]_{n \times n}, \quad A^+ = [a_{ij}]_{n \times n}. \quad (30)$$

Since $A^+ = \lim_{\varepsilon \rightarrow +0} Y(\varepsilon)$, then

$$a_{ij} = \lim_{\varepsilon \rightarrow +0} y_{ij}(\varepsilon), \quad i, j = 1, 2, \dots, n. \quad (31)$$

The entries of the last column of the matrix A^T are equal to zero (remind that $d_n = 0$). Hence, $y_{in}(\varepsilon) = 0$, $i = 1, 2, \dots, n$, and by this very fact

$$a_{in} = 0, \quad i = 1, 2, \dots, n. \quad (32)$$

From (30) it is seen that the entries of the matrix $Y(\varepsilon)$, for indices $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n-1$, are calculated by the rule

$$y_{ij}(\varepsilon) = x_{ij} d_j + x_{i, j+1} b_j. \quad (33)$$

The formulae (11) and (12) by which we calculate lower and upper triangular parts of the matrix $(A^T A + \varepsilon I)^{-1}$ respectively, are somewhat different. Therefore, for a fixed index j in the range $1 \leq j \leq n-1$ we consider separately two cases: $i = 1, 2, \dots, j$ and $i = j+1, j+2, \dots, n$.

- **Case $i = 1, 2, \dots, j$.**

Taking advantage of the expression (12) for the entries x_{ij} , from (33) we have

$$y_{ij}(\varepsilon) = t v_i (\mu_j d_j + \mu_{j+1} b_j). \quad (34)$$

Then, using the representations (18) for the quantities μ_i , we can write that

$$\begin{aligned} \mu_j d_j + \mu_{j+1} b_j &= (\overset{\circ}{\mu}_j + \gamma_j \varepsilon + O(\varepsilon^2)) d_j + (\overset{\circ}{\mu}_{j+1} + \gamma_{j+1} \varepsilon + O(\varepsilon^2)) b_j = \\ &= (\overset{\circ}{\mu}_j d_j + \overset{\circ}{\mu}_{j+1} b_j) + (\gamma_j d_j + \gamma_{j+1} b_j) \varepsilon + O(\varepsilon^2). \end{aligned}$$

As follows from the relation (23), $\overset{\circ}{\mu}_j d_j + \overset{\circ}{\mu}_{j+1} b_j = d_j (\overset{\circ}{\mu}_j + r_j \overset{\circ}{\mu}_{j+1}) = 0$.

Hence,

$$\mu_j d_j + \mu_{j+1} b_j = (\gamma_j d_j + \gamma_{j+1} b_j) \varepsilon + O(\varepsilon^2). \quad (35)$$

Substituting the expression (35) as well as the representations (24) and (29) of the quantities v_i and t respectively, into the right hand side of the equality (34) yields

$$y_{ij}(\varepsilon) = \frac{\overset{\circ}{v}_i (\gamma_j d_j + \gamma_{j+1} b_j) + O(\varepsilon)}{\overset{\circ}{\mu}_1 + d_1 (\gamma_1 d_1 + \gamma_2 b_1) + O(\varepsilon)}.$$

From here, by taking limit in the equality as $\varepsilon \rightarrow +0$ (see (31)), we find that

$$a_{ij} = \frac{\overset{\circ}{v}_i (\gamma_j d_j + \gamma_{j+1} b_j)}{\overset{\circ}{\mu}_1 + d_1 (\gamma_1 d_1 + \gamma_2 b_1)}, \quad i = 1, 2, \dots, j. \quad (36)$$

If we introduce the notation

$$u_j \equiv \gamma_j d_j + \gamma_{j+1} b_j, \quad j = 1, 2, \dots, n-1, \quad (37)$$

the entries a_{ij} from (36) may be written as follows:

$$a_{ij} = \frac{\overset{\circ}{v}_i u_j}{\overset{\circ}{\mu}_1 + d_1 u_1}, \quad i = 1, 2, \dots, j. \quad (38)$$

Using expressions (15), (16) and (20), one may show that the quantities u_j , defined in (37), satisfy the following relations:

$$u_{n-1} = -\frac{1}{b_{n-1}}, \quad u_j = -\frac{d_{j+1} u_{j+1} + \overset{\circ}{\mu}_{j+1}}{b_j}, \quad j = n-2, n-3, \dots, 1. \quad (39)$$

Moreover, it appears that the quantities u_j may be written as

$$u_j = \frac{(-1)^{n-j}}{b_j} \sum_{k=1}^{n-j} \left(\prod_{s=j+1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right), \quad j = 1, 2, \dots, n-1. \quad (40)$$

It can be proved by substituting the expression (40) into the relations (39).

Finally, let us replace the expression (40) for the quantities u_j as well as the expressions (22) and (27) for $\overset{\circ}{\mu}_i$ and $\overset{\circ}{v}_i$ respectively into (38). As a result, we obtain the following closed form expression for the entries of the upper triangular part of the matrix A^+ :

$$a_{ij} = \frac{(-1)^{i+j} \sum_{k=1}^{n-j} \left(\prod_{s=j+1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}{\prod_{s=1}^{i-1} r_s \cdot b_j \sum_{k=1}^n \left(\prod_{s=1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}, \quad i = 1, 2, \dots, j. \quad (41)$$

- **Case** $i = j+1, j+2, \dots, n$.

Using the expressions (11) for the entries x_{ij} , from (33) we have

$$y_{ij}(\varepsilon) = t \mu_i (v_j d_j + v_{j+1} b_j). \quad (42)$$

In accordance with representations (24), we write

$$\begin{aligned} v_j d_j + v_{j+1} b_j &= (\overset{\circ}{v}_j + \delta_j \varepsilon + O(\varepsilon^2)) d_j + (\overset{\circ}{v}_{j+1} + \delta_{j+1} \varepsilon + O(\varepsilon^2)) b_j = \\ &= (\overset{\circ}{v}_j d_j + \overset{\circ}{v}_{j+1} b_j) + (\delta_j d_j + \delta_{j+1} b_j) \varepsilon + O(\varepsilon^2). \end{aligned}$$

As follows from the relation (28), $\overset{\circ}{v}_j d_j + \overset{\circ}{v}_{j+1} b_j = d_j(\overset{\circ}{v}_j + r_j \overset{\circ}{v}_{j+1}) = 0$.

Hence,

$$v_j d_j + v_{j+1} b_j = (\delta_j d_j + \delta_{j+1} b_j) \varepsilon + O(\varepsilon^2). \quad (43)$$

If we substitute expression (43) as well as the representations (18) and (29) of the quantities μ_i and t respectively into the right hand side of the equality (42), we get

$$y_{ij}(\varepsilon) = \frac{\overset{\circ}{\mu}_i (\delta_j d_j + \delta_{j+1} b_j) + O(\varepsilon)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1) + O(\varepsilon)}.$$

By taking limit in the last equality as $\varepsilon \rightarrow +0$, we obtaine

$$a_{ij} = \frac{\overset{\circ}{\mu}_i (\delta_j d_j + \delta_{j+1} b_j)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1)}, \quad i = j + 1, \dots, n. \quad (44)$$

Similarly to the previous case, we introduce the notation

$$w_j \equiv \delta_j d_j + \delta_{j+1} b_j, \quad j = 1, 2, \dots, n - 1. \quad (45)$$

Then the entries a_{ij} from (44) may be rewritten by

$$a_{ij} = \frac{\overset{\circ}{\mu}_i w_j}{\overset{\circ}{\mu}_1 + d_1 u_1}, \quad i = j + 1, j + 2, \dots, n. \quad (46)$$

Having expressions (16), (17) and (26), we find that defined in (45) quantities w_j satisfy the following relations:

$$w_1 = -\frac{1}{d_1}, \quad w_j = -\frac{b_{j-1} w_{j-1} + \overset{\circ}{v}_j}{d_j}, \quad j = 2, 3, \dots, n - 1. \quad (47)$$

It turns out, however, that the quantities w_j may be written by

$$w_j = \frac{(-1)^j}{d_j} \sum_{k=1}^j \left(\prod_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod_{s=k}^{j-1} r_s \right), \quad j = 1, 2, \dots, n - 1. \quad (48)$$

It may readily make sure by substituting the expression (48) into the relations (47).

Now replace the expression (48) for the quantity w_j and the expression (22) of $\overset{\circ}{\mu}_i$ into the equality (46). Resulting formula for the entries of lower triangular part of the matrix A^+ is of the following type:

$$a_{ij} = \frac{(-1)^{i+j+1} \left(\prod_{s=i}^{n-1} r_s \right) \cdot \sum_{k=1}^j \left(\prod_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod_{s=k}^{j-1} r_s \right)}{d_j \sum_{k=1}^n \left(\prod_{s=1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}, \quad i = j + 1, \dots, n. \quad (49)$$

Thus, in (32), (41) and (49) we have got the entries of the matrix A^+ , expressed through the entries of the original bidiagonal matrix A . Summarizing the considerations of the section, we arrive at the following main statement.

Theorem. Let A be a bidiagonal matrix of the form (1), whose entries meet the conditions (2) and (3). Then the entries of the Moore–Penrose inverse matrix $A^+ = [a_{ij}]_{n \times n}$ are as follows:

a) for the indices $j = 1, 2, \dots, n-1$:

$$a_{ij} = \frac{(-1)^{i+j} \sum_{k=1}^{n-j} \left(\prod_{s=j+1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}{\prod_{s=1}^{i-1} r_s \cdot b_j \sum_{k=1}^n \left(\prod_{s=1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}, \quad i = 1, 2, \dots, j; \quad (50)$$

$$a_{ij} = \frac{(-1)^{i+j+1} \prod_{s=i}^{n-1} r_s \cdot \sum_{k=1}^j \left(\prod_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod_{s=k}^{j-1} r_s \right)}{d_j \sum_{k=1}^n \left(\prod_{s=1}^{n-k} \frac{1}{r_s} \right) \left(\prod_{s=n-k+1}^{n-1} r_s \right)}, \quad i = j+1, j+2, \dots, n, \quad (51)$$

where $r_s = b_s/d_s$, $s = 1, 2, \dots, n-1$ and $r_0 = r_n = 1$;

b) for the index $j = n$: $a_{in} = 0$, $i = 1, 2, \dots, n$. (52)

Below is an example to illustrate the Theorem.

Example. Consider the $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 1 \\ & & & & 0 \end{bmatrix}.$$

According to the Eqs. (50) and (51), the entries of the first $n-1$ columns of the matrix A^+ are as follows. For the indices $j = 1, 2, \dots, n-1$:

$$\begin{aligned} a_{ij} &= (-1)^{i+j} (1 - j/n), \quad i = 1, 2, \dots, j, \\ a_{ij} &= (-1)^{i+j+1} j/n, \quad i = j+1, j+2, \dots, n. \quad \diamond \end{aligned}$$

In the next section we discuss the issues connected with the practical computation of the matrix A^+ .

An Algorithm to Compute the Matrix A^+ . Obtained in the previous sections intermediate formulae and relations allow us to propose the following fairly simple algorithm for computing the entries of the matrix A^+ .

Algorithm 2d/pinv/special ($A \Rightarrow A^+$)

1. Compute the quantities r_s (see (21)):

$$r_s = b_s/d_s, \quad s = 1, 2, \dots, n-1.$$

2. Compute the quantities $\overset{\circ}{\mu}_i$ (see (19), (23)):

$$\overset{\circ}{\mu}_n = 1; \quad \overset{\circ}{\mu}_i = -r_i \overset{\circ}{\mu}_{i+1}, \quad i = n-1, n-2, \dots, 1.$$

3. Compute the quantities $\overset{\circ}{v}_i$ (see (25), (28)):

$$\overset{\circ}{v}_1 = 1; \quad \overset{\circ}{v}_{i+1} = -\overset{\circ}{v}_i / r_i, \quad i = 1, 2, \dots, n-1.$$

4. Compute the quantities u_j (see (39)):

$$u_{n-1} = -1/b_{n-1}; u_j = -(d_{j+1}u_{j+1} + \overset{\circ}{\mu}_{j+1})/b_j, \quad j = n-2, n-3, \dots, 1.$$

5. Compute the quantities w_j (see (47)):

$$w_1 = -1/d_1; w_j = -(b_{j-1}w_{j-1} + \overset{\circ}{v}_j)/d_j, j = 2, 3, \dots, n-1.$$

6. Set $a_{in} = 0, i = 1, 2, \dots, n$ (see (32)).

7. Compute the entries of upper triangular part of the matrix A^+ (see (38)):

$$a_{ij} = \overset{\circ}{v}_i u_j / (\overset{\circ}{\mu}_1 + d_1 u_1), i = 1, 2, \dots, j; j = 1, 2, \dots, n-1.$$

8. Compute the entries of lower triangular part of the matrix A^+ (see (46)):

$$a_{ij} = \overset{\circ}{\mu}_i w_j / (\overset{\circ}{\mu}_1 + d_1 u_1), i = j+1, j+2, \dots, n; j = 1, 2, \dots, n-1.$$

End.

Direct calculations show that the numerical implementation of the algorithm **2d/pinv/special** requires $O(n^2)$ arithmetical operations. By this very fact, the algorithm may be considered as an optimal one.

Concluding Remarks. In this paper we have deduced closed form expressions, as well as the numerical algorithm, to compute the entries of the Moore–Penrose inverse of bidiagonal matrices (1), under assumptions (2) and (3). On the base of obtained results in a subsequent paper we will consider bidiagonal matrices of the form (1) with any arrangement of one or more zeros on the main diagonal.

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