PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 1, p. 64–66

C O M M U N I C A T I O N S

M a t h e m a t i c s

ON THE ALMOST EVERYWHERE CONVERGENCE OF NEGATIVE ORDER CESARO MEANS OF FOURIER–WALSH SERIES

L. N. GALOYAN *, R. G. MELIKBEKYAN

Chair of Higher Mathematics, Faculty of Physics YSU, Armenia

In the paper is presented existence of an increasing sequence of natural numbers $M_\nu,\ \nu = 0,1,\dots$, such that for any $\varepsilon > 0$ there exists a measurable set E with a measure $\mu E > 1-\varepsilon$, such that for any function $f \in L^1[0,1]$ one can find a function $g \in L^1[0,1]$, which coincides with the function f on E, and for any $\alpha \neq -1,-2,\dots$ the Cesaro means $\sigma^\alpha_{M_\nu}(x,\tilde{f}), \nu = 0,1,\dots$, converges to $g(x)$ almost everywhere on [0,1].

MSC2010: Primary 42C10; Secondary 42B08.

***Keywords*:** Fourier–Walsh series, Cesaro means.

In the paper are investigated some questions of convergence of negative order Cesaro means of Fourier–Walsh series of corrected function. Let's recall the definition of Cesaro summation methods and Cesaro means for arbitrary numerical series. Let $S_n = u_0 + u_1 + \dots + u_n,\ n = 0,1,\dots$, be the partial sums of the series $\sum u_k$. For a fixed real number α with $\alpha \neq -1,-2,\dots$ the sequence

$$\sigma^\alpha_n(x) = \frac{1}{A^\alpha_n}\sum_{k=0}^{n} A^{\alpha-1}_{n-k}S_k = \frac{1}{A^\alpha_n}\sum_{k=0}^{n} A^\alpha_{n-k}u_k, \quad n = 0,1,\dots,$$

where $\{A^\alpha_n\}^\infty_{n=0}$ determined from the expression

$$\frac{1}{(1-z)^{1+\alpha}} = \sum_{n=0}^{\infty} A^\alpha_n z^n, \ \ |z| < 1,$$

* E-mail: levongaloyan@ysu.am

is called the (C,α) means or the α-order Cesaro means of the series $\sum u_k$. Further, we recall the definition of the system of Walsh–Paley functions $\{W_n(x)\}_{n=0}^{\infty}$ (see [2]). By definition $W_0(x)=1,\ x\in[0,1)$, and if

$$n=\sum_{s=1}^{k}2^{m_s},\ \ m_1>m_2>...>m_s,$$

the binary representation of the number n, then

$$W_n(x)=\prod_{s=1}^{k}R_{m_s}(x),$$

where $\{R_k(x)\}_{k=0}^{\infty}$ are Rademacher functions and are defined as follows:

$$R_0(x)=\begin{cases}1,\ \ x\in[0,1/2),\\ -1,\ \ x\in[1/2,1),\end{cases}$$

$$R_k(x)=R_0(2^k x),\ \ R_k(x+1)=R_k(x),\ \ k=1,2,...$$

In this note, we consider the a.e. convergence of negative order Cesaro means of Fourier–Walsh series in terms of classical theorems of correction. In this direction, important results relating to this paper were obtained in [1–4]. We present the following theorem.

Theorem. There exists an increasing sequence of natural numbers M_ν, $\nu=0,1,...$, such that for any $\varepsilon>0$ there exists a measurable set E with a measure $\mu E>1-\varepsilon$, such that for any function $f\in L^1[0,1]$ one can find a function $g\in L^1[0,1]$, which coincides with the function f on E, and for any $\alpha\neq-1,-2,...$ the Cesaro means $\sigma_{M_\nu}^{\alpha}(x,g),\nu=0,1,...$, converges to $g(x)$ almost everywhere on [0,1], where $\sigma_k^{\alpha}(x,g),\ k=0,1,...$, are the α-order Cesaro means of Fourier–Walsh series of function g.

Remark 1. Note that (see [5]) there exsists a function $f\in L^1[0,1]$ such that each subsequence of the sequence of Cesaro means of negative order of the Fourier–Walsh series of this function diverges almost everywhere. In the same paper it is established that for any $p\geq1$ there exists a function $f\in L^p[0,1]$, such that for any $\alpha<1/p-1$ and for any increasing sequence of natural numbers $\{m_k\}$, the Cesaro means $\sigma_{m_k}^{\alpha}(x,f)$ diverge in $L^p[0,1]$ norm.

Remark 2. The scheme of proof of Theorem is the following: for a given function $f\in L^1[0,1]$, the function g, which is obtained by modifying f on the set of "small measure", is represented in $L^1[0,1]$ by a series of nonintersecting polynomials in Walsh series $P_q(x)$ (which have small oscillations on the sets of "big measures") in the following way:

$$g(x)=\sum_{q=1}^{\infty}P_q(x),\ \ \text{where}\ \ P_q(x)=\sum_{r=m_q}^{\overline{m}_q}a_r^{(q)}W_r(x).$$

Further, it is proved that the series

$$\sum_{r=1}^{\infty}a_rW_r(x),\quad \text{where}\quad a_r=\begin{cases}a_r^{(q)},\ \ m_q\leq r\leq\overline{m}_q,\\ 0,\ \ \ \overline{m}_q<r<m_{q+1},\end{cases}$$

converges to the function g, and for previously constructed sequence M_q, $q = 0, 1, \ldots$ (which does not depend on the function f and on α) the relation

$$\sigma_{M_q}^{\alpha}(x, g) - S_{M_q}(x, g) \to 0 \text{ as } q \to \infty$$

holds a.e. on $[0, 1]$, where $S_k(x, g)$, $k = 0, 1, \ldots$, are partial sums of Fourier–Walsh series of function g.

The full proof of Theorem is voluminous, and we will not present it here.

Note that Theorem, in a less demanding formulation, is proved in [1] by Men'shov in the case of the trigonometric system. More precisely, the following fact was proved: for any $\alpha < 0$ with $\alpha \neq -1, -2, \ldots$, for each function $f \in L^1[-\pi, \pi]$ and for any perfect nowhere dense set $P \subset [-\pi, \pi]$, there exists a function $g \in L^1[-\pi, \pi]$ and an increasing sequence of natural numbers m_ν, $\nu = 0, 1, \ldots$, such that $f(x) = g(x)$ on P and $\lim\limits_{\nu \to \infty} \sigma_{m_\nu}^{\alpha}(x, g) = g(x)$, a.e. on $[-\pi, \pi]$, where $\sigma_k^{\alpha}(x, g)$, $k = 0, 1, \ldots$, are the α-order Cesaro means of trigonometric Fourier series of function g.

Note also that in [3] is proved the following generalisation of Men'shov theorem: for any perfect nowhere dense set $P \subset [-\pi, \pi]$, there exists an increasing sequence of natural numbers $M_\nu, \nu = 0, 1, \ldots$, such that for any function $f \in L^1[-\pi, \pi]$ there exists a function $g \in L^1[-\pi, \pi]$, such that $f(x) = g(x)$ on P and and for any $\alpha \neq -1, -2, \ldots$ the means $\sigma_{M_\nu}^{\alpha}(x, g), \nu = 0, 1, \ldots$, converges to $g(x)$ in $L^1[-\pi, \pi]$ metric and a.e. on $[-\pi, \pi]$.

Remark 3. In the last statement (see [3]) and in Theorem 1, by contrast with Men'shov theorem, the sequence $M_\nu, \nu = 0, 1, \ldots$, does not depend on the corrected function and on α.

Received 05.02.2016

REFERENCES

1. **Men'shov D.** Application of Cesaro Summability Methods of Negative Order to Trigonometric Fourier Series of Summable and Square-Summable Functions. // Mathematics of the USSR-Sbornik, 1974, v. 22, № 4, p. 497–515.
2. **Grigorian M.** On the Convergence of Fourier Series in the Metric of L^1. // Analysis Math., 1991, v. 17, p. 211–237.
3. **Galoyan L.N., Grigoryan M.G., Kobelyan A.Kh.** Convergence of Fourier Series in Classical Systems. // Mat. Sb., 2015, v. 206, № 7, p. 55–94.
4. **Grigoryan M.G., Galoyan L.N.** On the Uniform Convergence of Negative Order Cesaro Means of Fourier Series. // Journal of Mathem. Anal. and Appl., 2016, v. 434, p. 554–567. http://dx.doi.org/10.1016/j.jmaa.2015.09.001.
5. **Galoyan L.N.** On the Convergence in the Metric of L^p of Negative Order Cesaro Means of Fourier Series. // Izvestia NAN Armenii. Matematika, 2012, v. 47, № 3, p. 35–54.