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ON CANONICAL NOTION OF δ-REDUCTION AND ON TRANSLATION OF TYPED λ-TERMS INTO UNTYPED λ-TERMS

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In the paper typed and untyped λ-terms are considered. Typed λ-terms use variables of any order and constants of order ≤ 1. Constants of order 1 are strong computable functions with indeterminate values of arguments and every function has an untyped λ-term that λ-defines it. The so-called canonical notion of δ-reduction is introduced. This is the notion of δ-reduction that is used in the implementation of functional programming languages. For the canonical notion of δ-reduction the translation of typed λ-terms into untyped λ-terms is studied.

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Typed λ-terms, Canonical Notion of δ-Reduction. The definitions of this section can be found in [1–3]. A partially ordered set is said to be complete, if each of its linear ordered subsets has the least upper bound. It is easy to see that every complete set has a least element. Let A, B be nonempty partially ordered sets. A mapping $\varphi : A \to B$ is said to be monotonic, if $a \subseteq b$ implies $\varphi(a) \subseteq \varphi(b)$ for all $a, b \in A$ ($\subseteq$ is the symbol for partial ordering relation).

Let M be a partially ordered set, which has an element $\perp$, which corresponds to the indeterminate value. Each element of M is comparable with itself and with $\perp$, which is the least element of M. Let us define the set of types (denoted by $Types$).

1. $M \in Types$.
2. If $\beta, \alpha_1, \dots, \alpha_k \in Types$ $(k > 0)$, then the set of all monotonic mappings from $\alpha_1 \times \ldots \times \alpha_k$ into β (denoted by $[\alpha_1 \times \ldots \times \alpha_k \to \beta]$) belongs to $Types$.

If $\alpha \in Types$, then the order of type α (denoted by $ord(\alpha)$) will be a natural number, which is defined in the following way: if $\alpha = M$, then $ord(\alpha) = 0$, if $\alpha = [\alpha_1 \times ... \times \alpha_k \to \beta]$, where $\beta, \alpha_1, ... \alpha_k \in Types$, $k > 0$, then $ord(\alpha) = 1 + \max(ord(\alpha_1), \dots, ord(\alpha_k), ord(\beta))$, if x is a variable of type α and a constant $c \in \alpha$, then $ord(x) = ord(c) = ord(\alpha)$. Every type $\alpha \in Types$ is a complete set (see [1]).

Let $\alpha \in Types$ and V^T_α be a countable set of variables of type α, then $V^T = \bigcup\limits_{\alpha \in Types} V^T_\alpha$ is the set of all variables. The set of all terms, denoted by $\Lambda^T = \bigcup\limits_{\alpha \in Types} \Lambda^T_\alpha$, where Λ^T_α is the set of terms of type α, is defined in the following way:

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1. If $c \in \alpha$, $\alpha \in Types$, then $c \in \Lambda^T_\alpha$;
2. If $x \in V^T_\alpha$, $\alpha \in Types$, then $x \in \Lambda^T_\alpha$;
3. If $\tau \in \Lambda^T_{[\alpha_1 \times \ldots \times \alpha_k \to \beta]}$, $t_i \in \Lambda^T_{\alpha_i}$, where β, $\alpha_i \in Types$, $i = 1, \ldots, k$, $k \geq 1$, then $\tau(t_1, \ldots, t_k) \in \Lambda^T_\beta$ (the operation of application);
4. If $\tau \in \Lambda^T_\beta, x_i \in V^T_{\alpha_i}$, where $\beta, \alpha_i \in Types$, $i \neq j \Rightarrow x_i \neq x_j, i, j = 1, \ldots, k$, $k \geq 1$, then $\lambda x_1 \ldots x_k[\tau] \in \Lambda^T_{[\alpha_1 \times \ldots \times \alpha_k \to \beta]}$ (the operation of abstraction).

The notions of free and bound occurrences of variables in terms as well as the notion of a free variable are introduced in the ordinary way. The set of all free variables of a term t is denoted by $FV(t)$. A term which doesn't contain free variables is called a closed term. Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one term can be obtained from the other by renaming bound variables. In what follows congruent terms are considered identical.

Let $t \in \Lambda^T_\alpha$, $\alpha \in Types$ and $FV(t) \subset \{y_1, \ldots, y_n\}$, $\bar{y_0} = \langle y^0_1, \ldots, y^0_n \rangle$, where $y_i \in V^T_{\beta_i}$, $y^0_i \in \beta_i$, $\beta_i \in Types$, $i = 1, \ldots, n$, $n \geq 0$. The value of the term t for the values of the variables $y_1, \ldots, y_n$ equal to $\bar{y_0} = \langle y^0_1, \ldots, y^0_n \rangle$ is denoted by $Val_{\bar{y_0}}(t)$ and defined as follows:

1. If $t \equiv c$ and $c \in \alpha$, then $Val_{\bar{y_0}}(c) = c$;
2. If $t \equiv x$, $x \in V^T_\alpha$, then $Val_{\bar{y_0}}(x) = y^0_i$, where $FV(x) = \{x\} \subset \{y_1, \ldots, y_n\}$ and $x \equiv y_i$, $i = 1, \ldots, n$, $n \geq 1$;
3. If $t \equiv \tau(t_1, ..., t_k) \in \Lambda^T_\alpha$, where $\tau \in \Lambda^T_{[\alpha_1 \times \ldots \times \alpha_k \to \alpha]}, t_i \in \Lambda^T_{\alpha_i}, \alpha_i \in Types$, $i = 1, \ldots, k$, $k \geq 1$, then $Val_{\bar{y_0}}(\tau(t_1, \ldots, t_k)) = Val_{\bar{y_0}}(\tau)(Val_{\bar{y_0}}(t_1), \ldots, Val_{\bar{y_0}}(t_k))$;
4. If $t \equiv \lambda x_1 \ldots x_k[\tau] \in \Lambda^T_\alpha$, where $\alpha = [\alpha_1 \times \ldots \times \alpha_k \to \beta]$, $\tau \in \Lambda^T_\beta$, $x_i \in V^T_{\alpha_i}$, $\beta, \alpha_i \in Types$, $i = 1, \ldots, k$, $k \geq 1$, then $Val_{\bar{y_0}}(\lambda x_1 \ldots x_k[\tau]) \in [\alpha_1 \times \ldots \times \alpha_k \to \beta]$ and is defined as follows: let $\{y_1, \ldots, y_n\} \setminus \{x_1, \ldots, x_k\} = \{y_{j_1}, \ldots, y_{j_s}\}$, $s \geq 0$, and $\bar{z_0} = \langle y^0_{j_1}, \ldots, y^0_{j_s} \rangle$, then for any $\bar{x_0} = \langle x^0_1, \ldots, x^0_k \rangle$, where $x^0_i \in \alpha_i$, $i = 1, \ldots, k$, $Val_{\bar{y_0}}(\lambda x_1 \ldots x_k[\tau])(x^0_1, \ldots, x^0_k) = Val_{\bar{x_0}, \bar{z_0}}(\tau)$, where $\bar{x_0}, \bar{z_0} = \langle x^0_1, \ldots, x^0_k, y^0_{j_1}, \ldots, y^0_{j_s} \rangle$.

It follows from [1], that for any $\bar{y_0} = \langle y^0_1, \ldots, y^0_n \rangle$ and $\bar{y_1} = \langle y^1_1, \ldots, y^1_n \rangle$ such that $\bar{y_0} \subseteq \bar{y_1}$, where $y^0_i, y^1_i \in \beta_i$ $(1 \leq i \leq n)$, we have the following:

1. $Val_{\bar{y_0}}(t) \in \alpha$;
2. $Val_{\bar{y_0}}(t) \subseteq Val_{\bar{y_1}}(t)$.

Let terms $t_1, t_2 \in \Lambda^T_\alpha$, $\alpha \in Types$, $FV(t_1) \cup FV(t_2) = \{y_1, \ldots, y_n\}$, $y_i \in V^T_{\beta_i}$, $\beta_i \in Types$, $i = 1, \ldots, n$, $n \geq 0$, then terms t_1 and t_2 are called equivalent (denoted by $t_1 \sim t_2$), if for any $\bar{y_0} = < y^0_1, \ldots, y^0_n >$, where $y^0_i \in \beta_i$, $i = 1, \ldots, n$, we have the following: $Val_{\bar{y_0}}(t_1) = Val_{\bar{y_0}}(t_2)$. A term $t \in \Lambda^T_\alpha$, $\alpha \in Types$, is called a constant term with value $a \in \alpha$, if $t \sim a$.

Lemma 1. Let $t \in \Lambda^T_M$, $FV(t) = \{y_1, \ldots, y_n\}$, $y_i \in V^T_{\beta_i}$, $\beta_i \in Types$, $i = 1, \ldots, n$, $n \geq 0$, and for every $m \in M \setminus \{\perp\}$, $t \not\sim m$, then $Val_{\bar{\Omega}}(t) = \perp$, where $\bar{\Omega} = \langle \Omega_1, \ldots, \Omega_n \rangle$, Ω_i is the least element of the type β_i, $i = 1, \ldots, n$.

Proof. If $t \sim \perp$, then it is obvious that $Val_{\bar{\Omega}}(t) = \perp$. If $t \not\sim \perp$, then there exists such $m \in M \setminus \{\perp\}$ and $\bar{y_0} = \langle y^0_1, \ldots, y^0_n \rangle$, where $y^0_i \in \beta_i$, $i = 1, \ldots, n$, that $Val_{\bar{y_0}}(t) = m$. Since $\bar{\Omega} \subseteq \bar{y_0}$, we have $Val_{\bar{\Omega}}(t) \subseteq Val_{\bar{y_0}}(t)$ and from $t \not\sim m$ it follows that $Val_{\bar{\Omega}}(t) \neq m$. Thus we get $Val_{\bar{\Omega}}(t) = \perp$. $\square$

Further, is assumed that M is a recursive set and considered terms use variables of any order and constants of order ≤ 1, where the constants of order 1 are strong computable, monotonic functions with indeterminate values of arguments. A function $f : M^k \to M$, $k \geq 1$, with indeterminate values of arguments is said to be strong computable, if there exists an algorithm, which stops with value $f(m_1, \ldots, m_k)$ for all $m_1, \ldots, m_k \in M$ (see [2]). We suppose that each strong computable function with indeterminate values of arguments is given by its algorithm. We denote all such terms by Λ^T and denote all such terms of type α by Λ^T_α.

The notation $t[t_1,...,t_k]$ is used to show mutually different variables of interest $x_1,...,x_k$, $k \geq 1$, of a term t. The notation $t[t_1,\ldots,t_k]$ denotes the term obtained by the simultaneous substitution of the terms $t_1,\ldots,t_k$ for all free occurrences of variables $x_1,\ldots,x_k$ respectively, where $x_i \in V^T_{\alpha_i}$, $i \neq j \Rightarrow x_i \not\equiv x_j$, $t_i \in \Lambda^T_{\alpha_i}$, $\alpha_i \in Types$, $i,j = 1,\ldots,k$, $k \geq 1$. A substitution is said to be admissible, if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

A term $t \in \Lambda^T$ with a fixed occurrence of a subterm $\tau_1 \in \Lambda^T_\alpha$, where $\alpha \in Types$, is denoted by t_{τ_1} and a term with this occurrence of τ_1 replaced by τ_2, where $\tau_2 \in \Lambda^T_\alpha$ is denoted by t_{τ_2}.

A term of the form $\lambda x_1 \ldots x_k[\tau[x_1,\ldots,x_k]](t_1,\ldots,t_k)$, where $x_i \in V^T_{\alpha_i}$, $i \neq j \Rightarrow x_i \not\equiv x_j$, $\tau \in \Lambda^T$, $t_i \in \Lambda^T_{\alpha_i}$, $\alpha_i \in Types$, $i,j = 1,\ldots,k$, $k \geq 1$, is called a β-redex, its convolution is the term $\tau[t_1,\ldots,t_k]$. The set of all pairs (τ_0,τ_1), where τ_0 is a β-redex and τ_1 is its convolution, is called a notion of β-reduction and is denoted by β.

A term t_1 is said to be obtained from a term t_0 by one-step β-reduction (denoted by $t_0 \to_\beta t_1$), if $t_0 \equiv t_{\tau_0}$, $t_1 \equiv t_{\tau_1}, \tau_0$ is a β-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by β-reduction (denoted by $t_0 \to\to_\beta t$), if there exists a finite sequence of terms $t_1,\ldots,t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_\beta t_{i+1}$, where $i = 1,\ldots,n-1$.

A term containing no β-redexes is called a β-normal form. The set of all β-normal forms is denoted by $\beta - NF^T$. It follows from [4] that every term $t \in \Lambda^T$ is strongly β-normalized, i.e. every β-reduction chain for the term t has a finite length. Therefore, for every term $t \in \Lambda^T$ there exists a term $\tau \in \beta - NF^T$ such that $t \to\to_\beta \tau$.

δ-redex has a form $f(t_1,\ldots,t_k)$, where $f \in [M^k \to M]$, $t_i \in \Lambda^T_M$, $i = 1,\ldots,k$, $k \geq 1$, its convolution is either $m \in M$ and in this case $f(t_1,\ldots,t_k) \sim m$ or a subterm t_i and in this case $f(t_1,\ldots,t_k) \sim t_i$, $i = 1,\ldots,k$. A fixed set of term pairs (τ_0,τ_1), where τ_0 is a δ-redex and τ_1 is its convolution, is called a notion of δ-reduction and is denoted by δ.

A term t_1 is said to be obtained from a term t_0 by one-step δ-reduction (denoted by $t_0 \to_\delta t_1$), if $t_0 \equiv t_{\tau_0}$, $t_1 \equiv t_{\tau_1}, \tau_0$ is a δ-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by δ-reduction (denoted by $t_0 \to\to_\delta t$), if there exists a finite sequence of terms $t_1,\ldots,t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_\delta t_{i+1}$, where $i = 1,\ldots,n-1$.

A term containing no δ-redexes is called a δ-normal form. The set of all δ-normal forms is denoted by $\delta - NF^T$. It follows from [4], that every term $t \in \Lambda^T$ is strongly δ-normalized, i.e. every δ-reduction chain for the term t has a finite length. Therefore, for every term $t \in \Lambda^T$ there exists a term $\tau \in \delta - NF^T$ such that $t \to\to_\delta \tau$.

A term t_1 is said to be obtained from a term t_0 by one-step $\beta\delta$-reduction (denoted by $t_0 \to_{\beta\delta} t_1$), if either $t_0 \to_\beta t_1$ or $t_0 \to_\delta t_1$. A term t is said to be obtained from a term t_0 by $\beta\delta$-reduction (denoted by $t_0 \to\to_{\beta\delta} t$), if there exists a finite sequence of terms $t_1,\ldots,t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_{\beta\delta} t_{i+1}$, where $i = 1,\ldots,n-1$.

A term containing no $\beta\delta$-redexes is called a normal form. The set of all normal forms is denoted by NF^T. Due to [4] every term $t \in \Lambda^T$ is strongly $\beta\delta$-normalized, i.e. every $\beta\delta$-reduction chain for the term t has a finite length. Therefore, for every term $t \in \Lambda^T$ there exists a term $\tau \in NF^T$ such that $t \to\to_{\beta\delta} \tau$.

Note, that if $t_1 \to\to_{\beta\delta} t_2$, then $t_1 \sim t_2$, where $t_1,t_2 \in \Lambda^T_\alpha$, $\alpha \in Types$ [4].

A notion of δ-reduction is called a single-valued notion of δ-reduction if δ is a single-valued relation, i.e. if $\langle \tau_0,\tau_1 \rangle \in \delta$ and $\langle \tau_0,\tau_2 \rangle \in \delta$, then $\tau_1 \equiv \tau_2$, where $\tau_0,\tau_1,\tau_2 \in \Lambda^T_M$.

A notion of δ-reduction is called an effective notion of δ-reduction, if there exists an algorithm, which gives its convolution, if $f(t_1,\dots,t_k)$ is a δ-redex and stops with a negative answer otherwise for any term $f(t_1,\dots,t_k)$, where $f \in [M^k \to M]$, $t_i \in \Lambda_M^T$, $i=1,\dots,k$, $k \geq 1$.

Definition 1. An effective, single-valued notion of δ-reduction is called a canonical notion of δ-reduction, if:

1. $t \in \beta - NF^T$, $t \sim m$, $m \in M \setminus \{\perp\} \Rightarrow t \twoheadrightarrow_\delta m$;
2. $t \in \beta - NF^T$, $FV(t) = \emptyset$, $t \sim \perp \Rightarrow t \twoheadrightarrow_\delta \perp$.

If $t \in \beta - NF^T$, $t \sim m$, $m \in M \setminus \{\perp\}$, then $t \equiv m$ or $t \equiv f(t_1,\dots,t_k)$, where $t_i \in \Lambda_M^T$, $t_i \in \beta - NF^T$, $f \in [M^k \to M]$, $i=1,\dots,k$, $k \geq 1$. We introduce the notion of *rank* for such terms: $rank(m) = 0$, $rank(f(t_1,\dots,t_k)) = 1 + \max(rank(t_{j_1}),\dots,rank(t_{j_s}))$, where $t_{j_r} (1 \leq r \leq s, s \geq 0)$ is a constant term with value belonging to $M \setminus \{\perp\}$ and $t_{j_1},\dots,t_{j_s}$ is the maximal subsequence (of such terms) for the sequence $t_1,\dots,t_k$.

Theorem 1. *On Canonical Notion of δ-Reduction.* For every recursive set of strong computable, monotonic functions with indeterminate values of arguments there exists a canonical notion of δ-reduction.

Proof. Let C be a recursive set of strong computable, monotonic functions with indeterminate values of arguments. Now we define the notion of δ-reduction for the set C. For every function $f \in C$, where $f: M^k \to M, k \geq 1$, we have:

if $f(m_1,\dots,m_k) = m$, where $m, m_1,\dots,m_k \in M$, $m \neq \perp$, then $\langle f(\mu_1,\dots,\mu_k), m\rangle \in \delta$, where $\mu_i = m_i$ if $m_i \neq \perp$, and $\mu_i \equiv t_i$, $t_i \in \Lambda_M^T$ if $m_i = \perp$, $i=1,\dots,k$, $k \geq 1$;

if $f(m_1,\dots,m_k) = \perp$, where $m_1,\dots,m_k \in M$, then $\langle f(m_1,\dots,m_k), \perp\rangle \in \delta$.

Let us show that δ is a canonical notion of δ-reduction. It is easy to see that δ is an effective, single-valued notion of δ-reduction.

1. Let $t \in \beta - NF^T$, $t \sim m$, $m \in M \setminus \{\perp\}$. Let us show that $t \twoheadrightarrow_\delta m$. If $rank(t) = 0$, then $t \equiv m$, and $t \twoheadrightarrow_\delta m$. Let $rank(t) = n > 0$, then $t \equiv f(t_1,\dots,t_k)$, where $f \in C$ and $t_i \in \Lambda_M^T$, $t_i \in \beta - NF^T$, $i=1,\dots,k$, $k \geq 1$, and we suppose that Def. 1 (p. 1) holds for the terms of a *rank* $< n$. If $t_i \sim m_i, m_i \in M \setminus \{\perp\}$, then by the induction hypothesis, $t_i \twoheadrightarrow_\delta m_i$, $i=1,\dots,k$. Therefore $f(t_1,\dots,t_k) \twoheadrightarrow_\delta f(\tau_1,\dots,\tau_k)$, where $\tau_i = m_i$ if $t_i \sim m_i$, $m_i \in M \setminus \{\perp\}$, and $\tau_i \equiv t_i$, otherwise, $i=1,\dots,k$. According to the Lemma 1 we have $f(\mu_1,\dots,\mu_k) = m$, where $\mu_i \equiv \tau_i$ if $\tau_i = m_i$, $m_i \in M \setminus \{\perp\}$ and $\mu_i = \perp$ otherwise. Thus, $\langle f(\tau_1,\dots,\tau_k), m\rangle \in \delta$, $f(t_1,\dots,t_k) \twoheadrightarrow_\delta f(\tau_1,\dots,\tau_k) \to_\delta m$ and $t \twoheadrightarrow_\delta m$.

2. Let $t \in \beta - NF^T$, $FV(t) = \emptyset$, $t \sim \perp$. It is easy to see that in this case $t \twoheadrightarrow_\delta \perp$. $\square$

Theorem 2. Let δ be a canonical notion of δ-reduction, then:

1. $t \in \Lambda_M^T$, $t \sim m$, $m \in M \setminus \{\perp\} \Rightarrow t \twoheadrightarrow_{\beta\delta} m$;
2. $t \in \Lambda_M^T$, $FV(t) = \emptyset$, $t \sim \perp \Rightarrow t \twoheadrightarrow_{\beta\delta} \perp$.

Proof.

1) Let $t \in \Lambda_M^T$, $t \sim m$, $m \in M \setminus \{\perp\}$, then, according to [4], there exists a term $\tau \in \beta - NF^T$ such that $t \twoheadrightarrow_\beta \tau$ and $\tau \sim m$. Therefore, according to the Def. 1 (p. 1) we have $\tau \twoheadrightarrow_\delta m$, and $t \twoheadrightarrow_{\beta\delta} m$.

2) If $t \in \Lambda_M^T$, $FV(t) = \emptyset$, $t \sim \perp$, then, according to [4], there exists a term $\tau \in \beta - NF^T$, $FV(\tau) = \emptyset$ such that $t \twoheadrightarrow_\beta \tau$ and $\tau \sim \perp$. Therefore, according to the Def. 1 (p. 2) we have $\tau \twoheadrightarrow_\delta \perp$, and $t \twoheadrightarrow_{\beta\delta} \perp$. $\square$

Lemma 2. Let δ be a canonical notion of δ-reduction, let $t[x] \in \Lambda_M^T$, $x \in V_\alpha^T$, $\alpha \in Types$ and $t[x] \twoheadrightarrow_{\beta\delta} m$, $m \in M \setminus \{\perp\}$. Then for any $\tau \in \Lambda_\alpha^T$ we have $t[\tau] \twoheadrightarrow_{\beta\delta} m$.

Proof. $\lambda x[t[x]](\tau) \twoheadrightarrow_{\beta\delta} \lambda x[m](\tau) \to_\beta m$, therefore, according to [4] we have $\lambda x[t[x]](\tau) \sim m$. On the other hand, $\lambda x[t[x]](\tau) \to_\beta t[\tau]$ and according to [4] $t[\tau] \sim m$ and so from the p. 1 of Theorem 2, we get $t[\tau] \twoheadrightarrow_{\beta\delta} m$. $\square$

Untyped λ-Terms. The definitions of this section can be found in [3,5]. Let us fix a countable set of variables V. The set of terms Λ is defined as follows:

1. if $x \in V$, then $x \in \Lambda$;
2. if $t_1, t_2 \in \Lambda$, then $(t_1 t_2) \in \Lambda$ (the operation of application);
3. if $x \in V$ and $t \in \Lambda$, then $(\lambda x t) \in \Lambda$ (the operation of abstraction).

Let us introduce the following shorthand notations: a term $(\ldots(t_1 t_2)\ldots t_k)$, where $t_i \in \Lambda$, $i = 1,\ldots,k$, $k > 1$, is denoted by $t_1 t_2 \ldots t_k$ and a term $(\lambda x_1 (\lambda x_2(\ldots(\lambda x_n t)\ldots)))$, where $x_j \in V$, $j = 1,\ldots,n$, $n > 0$, $t \in \Lambda$, is denoted by $\lambda x_1 x_2 \ldots x_n.t$.

The notions of free and bound occurrences of variables in terms as well as the notion of free variable are introduced in the conventional way. The set of all free variables of a term t is denoted by $FV(t)$. A term, which doesn't contain free variables, is called a closed term. Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one term can be obtained from the other by renaming bound variables. In what follows congruent terms are considered identical.

The notation $t[x_1,\ldots,x_k]$ is used to show mutually different variables of interest $x_1,\ldots,x_k$, $k \geq 1$, of a term t. The notation $t[t_1,\ldots,t_k]$ denotes the term obtained by the simultaneous substitution of the terms $t_1,\ldots,t_k$ for all free occurrences of variables $x_1,\ldots,x_k$ respectively, $i \neq j \Rightarrow x_i \not\equiv x_j$, $i,j = 1,\ldots,k$, $k \geq 1$. A substitution is said to be admissible, if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

A term t with a fixed occurrence of a subterm τ_1 is denoted by t_{τ_1}, and a term with this occurrence of τ_1 replaced by a term τ_2 is denoted by t_{τ_2}.

A term of the form $(\lambda x.t[x])\tau$ is called a β-redex and the term $t[\tau]$ is called its convolution. The set of all term pairs (τ_0, τ_1), where τ_0 is a β-redex and τ_1 is its convolution, is called a notion of β-reduction and is denoted by β.

A term t_1 is said to be obtained from a term t_0 by one-step β-reduction (denoted by $t_0 \to_\beta t_1$) if $t_0 \equiv t_{\tau_0}$, $t_1 \equiv t_{\tau_1}$, τ_0 is a β-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by β-reduction (denoted by $t_0 \twoheadrightarrow_\beta t$), if there exists a finite sequence of terms $t_1,\ldots,t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_\beta t_{i+1}$, where $i = 1,\ldots,n-1$.

A term containing no β-redexes is called a normal form. The set of all normal forms is denoted by NF and the set of all closed normal forms is denoted by NF^0. A term t is said to have a normal form, if there exists a term τ such that $\tau \in NF$ and $t \twoheadrightarrow_\beta \tau$. From the Church–Rosser theorem [5] it follows, that if $t \twoheadrightarrow_\beta \tau_1, t \twoheadrightarrow_\beta \tau_2$, $\tau_1, \tau_2 \in NF$, then $\tau_1 \equiv \tau_2$.

If a term has a form $\lambda x_1 \ldots x_k.x t_1 \ldots t_n$, where $x_1,\ldots,x_k$, $x \in V$, $t_1,\ldots,t_n \in \Lambda$, $k,n \geq 0$, it is called a head normal form and x is called its head variable. The set of all head normal forms is denoted by HNF. A term t is said to have a head normal form, if there exists a term τ such that $\tau \in HNF$ and $t \twoheadrightarrow_\beta \tau$. It is known that $NF \subset HNF$, but $HNF \not\subset NF$ (see [5]).

Let $\lambda x_1 \ldots x_k.((\lambda x.t)\tau)t_1 \ldots t_n$ be a term, where $x_1,\ldots,x_k$, $x \in V$, $t_1,\ldots,t_n$, $\tau \in \Lambda$, $k,n \geq 0$, then the β-redex $(\lambda x.t)\tau$ is called a head β-redex. Obviously, every head β-redex of the term is its left β-redex, but not every left β-redex of the term is its head redex. Recall that if a term has a head normal form, then the reducing chain, where always the head β-redex is chosen, leads to a head normal form and if a term has a normal form, then the reducing chain, where always the leftmost β-redex is chosen, leads to the normal form [5].

Lemma 3. [3]. Let t_μ be a term with a fixed occurrence of a term μ, which doesn't have a head normal form, and let ν be any term, then:

1. $t_\mu \twoheadrightarrow_\beta \tau$, where $\tau \in NF \Rightarrow t_\nu \twoheadrightarrow_\beta \tau$;
2. t_μ has a head normal form $\Rightarrow t_\nu$ has a head normal form.

Translation. Let M be a recursive, partially ordered set, which has a least element $\perp$ and every element of M is comparable with itself and with $\perp$. Every $m \in M$ is mapped to an untyped term in the following way:

if $m \in M \setminus \{\perp\}$, then $m' \in NF^0$ and for any $m_1, m_2 \in M \setminus \{\perp\}, m_1 \neq m_2 \Rightarrow m_1' \not\equiv m_2'$;

if $m \equiv \perp$, then $m' \equiv \Omega \equiv (\lambda x.xx)(\lambda x.xx), x \in V$.

We say that a term Φ λ-defines (see [2]) the function $f : M^k \to M, k \geq 1$, with indeterminate values of arguments, if for any $m_1, \ldots, m_k \in M$ we have:

$f(m_1, \ldots, m_k) = m \neq \perp \Rightarrow \Phi m_1' \ldots m_k' \to\to_\beta m'$;

$f(m_1, \ldots, m_k) = \perp \Rightarrow \Phi m_1' \ldots m_k'$doesn't have a head normal form.

We consider typed terms using functions from a recursive set C. Every $f \in C$ is a strong computable function with intermediate values of arguments, which has an untyped λ-term that λ-defines it. From [2] it follows that every $f \in C$ is a strong computable, monotonic function with indeterminate values of arguments. Thus, according to the Theorem 1, there exists a canonical notion of δ-reduction for the set C. Let us consider the algorithm of translation of any typed term t to untyped term t' studied in [3]:

if $t \equiv m \in M$, then $t' \equiv m'$;

if $t \in C$, then $FV(t') = \emptyset$ and t' λ-defines t;

if $t \equiv x \in V^T$, then $x' \in V$ and for any $x_1, x_2 \in V^T$, $x_1 \not\equiv x_2 \Rightarrow x_1' \not\equiv x_2'$;

if $t \equiv \tau(t_1, \ldots, t_k)$, $k \geq 1$, then $t' \equiv \tau' t_1' \ldots t_k'$;

if $t \equiv \lambda x_1 \ldots x_n[\tau]$, $n \geq 1$, then $t' \equiv \lambda x_1' \ldots x_n'.\tau'$.

Lemma 4. [3]. $t, \tau \in \Lambda^T, t \to\to_\beta \tau \Rightarrow t' \to\to_\beta \tau'$.

Lemma 5. Let δ be a canonical notion of δ-reduction, then:

1. $t \in \beta - NF^T, t \to\to_\delta m, m \in M \setminus \{\perp\} \Rightarrow t' \to\to_\beta m'$,

2. $t \in \beta - NF^T, FV(t) = \emptyset, t \to\to_\delta \perp \Rightarrow t'$ does not have a head normal form.

Proof.

1) Let $t \in \beta - NF^T$, $t \to\to_\delta m$, $m \in M \setminus \{\perp\}$. We have two cases: 1a) $FV(t) = \emptyset$, and 1b) $FV(t) \neq \emptyset$.

1a) Let $t \in \beta - NF^T$, $FV(t) = \emptyset$, $t \to\to_\delta m$, $m \in M \setminus \{\perp\}$. We will show that $t' \to\to_\beta m'$. Let $t \to_\delta t_1 \to_\delta \ldots \to_\delta t_n \equiv m$, where $t_i \in \beta - NF^T$, $FV(t_i) = \emptyset$, $i = 1, \ldots, n$, $n \geq 0$. If $n = 0$, then $t \equiv m$, $t' \equiv m'$ and $t' \to\to_\beta m'$. Let $n > 0$ and suppose that (1a) holds for $n - 1$. Let $t \equiv t_{f(m_1,\ldots,m_k)} \to_\delta t_{m_0} \equiv t_1$, where $\langle f(m_1, \ldots, m_k), m_0 \rangle \in \delta$, $m_0, m_1, \ldots, m_k \in M$, $k \geq 1$.

If $m_0 \neq \perp$, then $f(m_1, \ldots, m_k) = m_0 \Rightarrow f' m_1' \ldots m_k' \to\to_\beta m_0'$ and $t' \equiv t'_{f' m_1' \ldots m_k'} \to\to_\beta t'_{m_0'} \equiv t_1'$. Since $t_1 \to_\delta \ldots \to_\delta t_n \equiv m$, by induction hypothesis we get $t_1' \to\to_\beta m'$. Therefore, $t' \to\to_\beta m'$.

If $m_0 = \perp$, then $t_1 \equiv t_\perp$ and $t_1' \equiv t'_\Omega$. Since $t_1 \to_\delta \ldots \to_\delta t_n \equiv m$, by induction hypothesis we get, $t_1' \to\to_\beta m'$. Therefore according to the p. 1 of Lemma 3, $t' \equiv t'_{f' m_1' \ldots m_k'} \to\to_\beta m'$.

1b) Let $t \in \beta - NF^T$, $FV(t) \neq \emptyset, t \to\to_\delta m$, $m \in M \setminus \{\perp\}$. We will show that $t' \to\to_\beta m'$. Let $FV(t) = \{x_1, \ldots, x_s\}$, $x_i \in V_{\alpha_i}$, $\alpha_i \in Types$, $i = 1, \ldots, s$, $s \geq 1$. Let $\Omega_i \in \Lambda_{\alpha_i}$ be a term that represents the least element of type α_i obtained by the operation of abstraction and term $\perp$, $i = 1, \ldots, s$. One can show that Ω_i' does not have a head normal form, $i = 1, \ldots, s$. Since $t \equiv t[x_1, \ldots, x_s] \to\to_\delta m$, then according to [4] $t[x_1, \ldots, x_s] \sim m$, therefore, $t[\Omega_1, \ldots, \Omega_s] \sim m$ and according to the p. 1 of Theorem 2 we get $t[\Omega_1, \ldots, \Omega_s] \to\to_{\beta\delta} m$. According to [4] there exists a term $\tau \in \beta - NF^T$, $FV(\tau) = \emptyset$, such that $t[\Omega_1, \ldots, \Omega_s] \to\to_\beta \tau$ and $\tau \sim m$, then by Def. 1 (p. 1), we conclude that $\tau \to\to_\delta m$. From Lemma 4 it follows that, $t'[\Omega_1', \ldots, \Omega_s'] \to\to_\beta \tau'$, by the case (1a) we get $\tau' \to\to_\beta m'$. So, $t'[\Omega_1', \ldots, \Omega_s'] \to\to_\beta m'$ and, according to the p. 1 of Lemma 3, $t' \equiv t'[x_1', \ldots, x_s'] \to\to_\beta m'$.

2) Let $t \in \beta - NF^T$, $FV(t) = \emptyset$, $t \twoheadrightarrow_\delta \perp$. We will show that t' does not have a head normal form. Let $t \rightarrow_\delta t_1 \rightarrow_\delta \ldots \rightarrow_\delta t_n \equiv \perp$, $t_i \in \beta - NF^T$, $FV(t_i) = \emptyset$, $i = 1, \ldots, n$, $n \geq 0$. If $n = 0$, then $t \equiv \perp$, $t' \equiv \Omega$ and t' does not have a head normal form. Let $n > 0$ and suppose that (2) holds for $n - 1$. Let $t \equiv t_{f(m_1,\ldots,m_k)} \rightarrow_\delta t_{m_0} \equiv t_1$, where $\langle f(m_1, \ldots, m_k), m_0 \rangle \in \delta$, $m_0, m_1, \ldots, m_k \in M$, $k \geq 1$.

If $m_0 \neq \perp$, then $f(m_1, \ldots, m_k) = m_0 \Rightarrow f'm'_1 \ldots m'_k \twoheadrightarrow_\beta m'_0$, $t' \equiv t'_{f'm'_1\ldots m'_k} \twoheadrightarrow_\beta t'_{m'_0} \equiv t'_1$. Since $t_1 \rightarrow_\delta \ldots \rightarrow_\delta t_n \equiv \perp$, by the induction hypothesis t'_1 does not have a head normal form. Therefore, t' does not have a head normal form too.

If $m_0 = \perp$, then $t_1 \equiv t_\perp$ and $t'_1 \equiv t'_\Omega$. Since $t_1 \rightarrow_\delta \ldots \rightarrow_\delta t_n \equiv \perp$, by induction hypothesis t'_1 does not have a head normal form. Since $f(m_1, \ldots, m_k) = \perp$, $f'm'_1 \ldots m'_k$ does not have a head normal form. Therefore according to p. 2 of Lemma 3, $t' \equiv t'_{f'm'_1\ldots m'_k}$ does not have a head normal form too. □

Now we give an example of canonical notion of δ-reduction and a term t such that $t \in \beta - NF^T$, $FV(t) \neq \emptyset$, $t \twoheadrightarrow_\delta \perp$ and term t' has a head normal form. Let $M = N \cup \{\perp\}$, where $N = \{0, 1, 2, \ldots\}$ and suppose that $C = \{f\}$, where $f : M \to M$ satisfies $f(m) = \perp$ for any $m \in M$ and so $f \in [M \to M]$. Let $\delta = \{\langle f(\tau), \perp \rangle | \tau \in \Lambda^T_M\}$, it is easy to see that δ is the canonical notion of δ-reduction. Let $\perp' \equiv \Omega, 0' \equiv I, (n+1)' \equiv \lambda x.xFn', f' \equiv \lambda x.x\Omega$, where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, $I \equiv \lambda x.x$, $F \equiv \lambda xy.y$, x, $y \in V$. It is easy to see that the term $f' \in \Lambda$ λ-defines the function f. Let $t \equiv f(z)$, where $z \in V^T_M$. It is easy to see that $f(z) \in \beta - NF^T$, $FV(f(z)) = \{z\} \neq \emptyset$, $f(z) \rightarrow_\delta \perp$, therefore, $f(z) \twoheadrightarrow_\delta \perp$ and $t' \equiv f'z' \equiv (\lambda x.x\Omega)z' \rightarrow_\beta z'\Omega$, where $z' \in V$ and $z'\Omega$ is a head normal form.

Theorem 3. *On Translation*. Let δ be a canonical notion of δ-reduction, then:

1. $t \in \Lambda^T_M$, $t \twoheadrightarrow_{\beta\delta} m$, $m \in M \setminus \{\perp\} \Rightarrow t' \twoheadrightarrow_\beta m'$;
2. $t \in \Lambda^T_M$, $FV(t) = \emptyset$, $t \twoheadrightarrow_{\beta\delta} \perp \Rightarrow t'$ does not have a head normal form.

Proof.

1) Let $t \in \Lambda^T_M$, $t \twoheadrightarrow_{\beta\delta} m$, $m \in M \setminus \{\perp\}$. According to [4], there exists a term $\tau \in \beta - NF^T$ such that $t \twoheadrightarrow_\beta \tau$ and $\tau \sim m$. From Lemma 4 it follows that $t' \twoheadrightarrow_\beta \tau'$. By Def. 1 (p. 1), $\tau \twoheadrightarrow_\delta m$, and, according to the p. 1 of Lemma 5, we get $\tau' \twoheadrightarrow_\beta m'$. Therefore $t' \twoheadrightarrow_\beta m'$.

2) Let $t \in \Lambda^T_M$, $FV(t) = \emptyset$, $t \twoheadrightarrow_{\beta\delta} \perp$. According to [4], there exists a term $\tau \in \beta - NF^T$, $FV(\tau) = \emptyset$ such that $t \twoheadrightarrow_\beta \tau$ and $\tau \sim \perp$. According to Lemma 4, $t' \twoheadrightarrow_\beta \tau'$, by Def. 1 (p. 2), $\tau \twoheadrightarrow_\delta \perp$, p. 2 of Lemma 5 implies that τ' does not have a head normal form. So, t' does not have a head normal form too. □

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