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M a t h e m a t i c s

GEOMETRIC PROBABILITY CALCULATION FOR A TRIANGLE

N. G. AHARONYAN *, H. O. HARUTYUNYAN **

Chair of the Theory of Probability and Mathematical Statistics YSU, Armenia

Let $P(L(\omega)\subset\mathbf{D})$ is the probability that a random segment of length l in $\mathbb{R}^n$ having a common point with body $\mathbf{D}$ entirely lies in $\mathbf{D}$. In the paper, using a relationship between $P(L(\omega)\subset\mathbf{D})$ and covariogram of $\mathbf{D}$ the explicit form of $P(L(\omega)\subset\mathbf{D})$ for arbitrary triangle on the plane is obtained.

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Introduction. Let $\mathbb{R}^n$ $(n\geq 2)$ be the n-dimensional Euclidean space, $\mathbf{D}\subset\mathbb{R}^n$ be a bounded convex body with inner points, and V_n be the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

Consider the set of the segments of a constant length that are contained in $\mathbf{D}$. The measure evaluation problem of such segment sets no simple solution and depends on the shape of $\mathbf{D}$. It is known the explicit form for the kinematic measures of the disk, the rectangle, if the length of the segment is less than the smaller side of the rectangle (see [1, 2]), the equilateral triangle, the rectangle and the regular pentagon (for an arbitrary length of the segment) [3].

D e f i n i t i o n 1. (see [2]). The function

$$C(\mathbf{D},h)=V_n(\mathbf{D}\cap(\mathbf{D}+h)),\quad h\in\mathbb{R}^n,$$

is called the covariogram of the body $\mathbf{D}$. Here $\mathbf{D}+h=\{x+h,x\in\mathbf{D}\}$.

Let S^{n-1} denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ centered at the origin. We consider a random line, which is parallel to $\mathbf{u}\in S^{n-1}$ and intersects $\mathbf{D}$, that is, an element from the set:

$$\Omega_1(\mathbf{u})=\{\text{lines, which are parallel to }\mathbf{u}\text{ and intersect }\mathbf{D}\}.$$

Let $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$ be the orthogonal projection of $\mathbf{D}$ onto the hyperplane $\mathbf{u}^\perp$ (here $\mathbf{u}^\perp$ stands for the hyperplane with normal $\mathbf{u}$ and passing through the origin).

* E-mail: narine78@ysu.am **E-mail: harutyunyan.hripsime@ysu.am

A random line, which is parallel to $\mathbf{u}$ and intersects $\mathbf{D}$, has an intersection point (denoted by x) with $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$. We can identify the points of $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$ and the lines, which intersect $\mathbf{D}$ and are parallel to $\mathbf{u}$, meaning that we can identify the sets $\Omega_1(\mathbf{u})$ and $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$. Assuming that the intersection point x is uniformly distributed over the convex body $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$, we can define the following distribution function.

D e f i n i t i o n 2. The function

$$F(\mathbf{u},t) = \frac{V_{n-1}\{x \in \Pi r_{\mathbf{u}^\perp}\mathbf{D} : V_1(g(\mathbf{u},x)\cap\mathbf{D}) < t)\}}{b_{\mathbf{D}}(\mathbf{u})}$$

is called orientation-dependent chord length distribution function of $\mathbf{D}$ in direction $\mathbf{u}$ at a point $t \in \mathbb{R}^1$, where $g(\mathbf{u},x)$ is the line, which is parallel to $\mathbf{u}$ and intersects $\Pi r_{\mathbf{u}^\perp}\mathbf{D}$ at the point x and $b_{\mathbf{D}}(\mathbf{u}) = V_{n-1}(\Pi r_{\mathbf{u}^\perp}\mathbf{D})$.

Observe that each vector $h \in \mathbb{R}^n$ can be represented in the form $h = (\mathbf{u},t)$, where $\mathbf{u}$ is the direction of h, and t is the length of h.

Let $L(\omega)$ be a random segment of length $l > 0$, which is parallel to a given fixed direction $\mathbf{u} \in S^{n-1}$ and intersects $\mathbf{D}$. Consider the random variable $|L|(\omega) := V_1(L(\omega)\cap\mathbf{D})$, where $L(\omega) \in \Omega_2(\mathbf{u})$, and the set $\Omega_2(\mathbf{u})$ is defined as follows:

$$\Omega_2(\mathbf{u}) = \{\text{segments of lengths } l\text{, which are parallel to } \mathbf{u} \text{ and intersect } \mathbf{D}\}.$$

Observe that each random segment $L(\omega)$ lying on a line $g(\mathbf{u},x)$ can be specified by the coordinates $(g(\mathbf{u},x),y)$, where y is the one-dimensional coordinate of the center of $L(\omega)$ on the line $g(\mathbf{u},x)$. As the origin on the line $g(\mathbf{u},x)$ we take one of the intersection points of the line $g(\mathbf{u},x)$ with the boundary of domain $\mathbf{D}$. Using the above notation, we can identify $\Omega_2(\mathbf{u})$ with the following set:

$$\Omega_2(\mathbf{u}) = \left\{(x,y) : x \in \Pi r_{\mathbf{u}^\perp}\mathbf{D}, \quad y \in \left[-\frac{l}{2}, \chi(\mathbf{u},x) + \frac{l}{2}\right]\right\},$$

where $\chi(\mathbf{u},x) = V_1(g(\mathbf{u},x)\cap\mathbf{D})$. Note that the set $\Omega_2(\mathbf{u})$ does not depend on the choice of the origin of the line $g(\mathbf{u},x)$, and the choice of the positive direction follows from the explicit form of the range of y. Further, we set

$$B_{\mathbf{D}}^{\mathbf{u},t} = \{(x,y) \in \Omega_2(\mathbf{u}) : \ |L|(x,y) < t\}, \quad t \in \mathbb{R}^1,$$

and observe that the sets $\Omega_2(\mathbf{u})$ and $B_{\mathbf{D}}^{\mathbf{u},t}$ are measurable subsets of $\mathbb{R}^n$.

D e f i n i t i o n 3. The function

$$F_{|L|}(\mathbf{u},t) = \frac{V_n(B_{\mathbf{D}}^{\mathbf{u},t})}{V_n(\Omega_2(\mathbf{u}))} = \frac{1}{V_n(\Omega_2(\mathbf{u}))}\int_{B_{\mathbf{D}}^{u,t}} dx\,dy$$

is called orientation-dependent distribution function of the length of a random segment L in direction $\mathbf{u} \in S^{n-1}$.

Let G_n be the space of all lines g in $\mathbb{R}^n$. A line $g \in G_n$ can be specified by its direction $\mathbf{u} \in S^{n-1}$ and its intersection point x in the hyperplane $\mathbf{u}^\perp$. The density $d\mathbf{u}^\perp$ is the volume element $d\mathbf{u}$ of the unit sphere S^{n-1}, and dx is the volume element on $\mathbf{u}^\perp$ at x. Let $\mu(\cdot)$ be a locally finite measure on G_n, invariant under the group of Euclidian motions. It is well known that the element of $\mu(\cdot)$ up to a constant factor has the following form (see [1]):

$$\mu(dg) = dg = d\mathbf{u}\,dx.$$

Denote by $O_{n-1} = \sigma_{n-1}(S^{n-1})$ the surface area of the unit sphere in $\mathbb{R}^n$. For each bounded convex body $\mathbf{D}$, we denote the set of lines that intersect $\mathbf{D}$ by

$$[\mathbf{D}] = \{g \in G_n,\ g \cap \mathbf{D} \neq \emptyset\}.$$

We have (see [1])

$$\mu([\mathbf{D}]) = \frac{O_{n-2}V_{n-1}(\partial \mathbf{D})}{2(n-1)}.$$

A random line in $[\mathbf{D}]$ is the one with distribution proportional to the restriction of μ to $[\mathbf{D}]$. Therefore, for any $t \in \mathbb{R}^1$ we have

$$F(t) = \frac{\mu(\{g \in [\mathbf{D}],\ V_1(g \cap \mathbf{D}) < t\})}{\mu([\mathbf{D}])},$$

which is called the chord length distribution function of $\mathbf{D}$. Let L be a random segment of length l in $\mathbb{R}^n$ and let $K(\cdot)$ be the kinematic measure of L [1]. If $g \in G_n$ is the line containing L and y is the one-dimensional coordinate of the center of L on the line g, then the element of the kinematic measure up to a constant factor is given by

$$dK = dg\, dy\, dK_{[1]},$$

where dy is the one-dimensional Lebesgue measure on g and $dK_{[1]}$ is a motion element in $\mathbb{R}^n$ that leaves g unchanged (see [1,4–7]).

Note that in the case, where the segment is orientated, the constant factor is equal to 1, while for the unoriented segment it is equal to $1/2$. In this paper we consider only the case of unoriented segments. The length $|L|$ of a random segment L, provided that it hits the body $\mathbf{D}$, has the following distribution function:

$$F_{|L|}(t) = \frac{K(L : L \cap \mathbf{D} \neq \emptyset, V_1(L \cap \mathbf{D}) < t)}{K(L : L \cap \mathbf{D} \neq \emptyset)}, \quad t \in \mathbb{R}^1.$$

Denote by $\mathbf{P}(L(\omega) \subset \mathbf{D})$ probability, that random segment of length l in $\mathbb{R}^n$ having a common point with body $\mathbf{D}$ entirely lying in body $\mathbf{D}$ (in this case the direction of the segment $L(\omega)$ is arbitrary).

Proposition (see [7]). Probability $\mathbf{P}(L(\omega) \subset \mathbf{D})$ in terms of chord length distribution function $F(t)$ has the following form:

$$\mathbf{P}(L(\omega) \subset \mathbf{D}) = \frac{O_{n-2}V_{n-1}(\partial \mathbf{D})\left(\int_0^l F(z)dz - l\right) + (n-1)O_{n-1}V_n(\mathbf{D})}{(n-1)O_{n-1}V_n(\mathbf{D}) + lO_{n-2}V_{n-1}(\partial \mathbf{D})}.$$

Case of a Triangle. For any body $\mathbf{D}$ of the $\mathbb{R}^n$ we have (see [7])

$$\mathbf{P}(L(\omega) \subset \mathbf{D}) = \frac{1}{O_{n-1}} \int_{S^{n-1}} \frac{C(\mathbf{D}, \mathbf{u}, l)}{V_n(D) + l \cdot b_D(\mathbf{u})}\, d\mathbf{u},$$

while the kinematic measure of the segments entire lying in $\mathbf{D}$ is calculated by the following formula:

$$K(L(\omega) \subset \mathbf{D}) = \int_{S^{n-1}} C(\mathbf{D}, \mathbf{u}, l)\, d\mathbf{u}.$$

For any planar bounded convex domain we have

$$\mathbf{P}(L(\omega) \subset \mathbf{D}) = \frac{1}{\pi S(D) + l|\partial D|} \int_0^{\pi} C(D, u, l)\, du. \quad (1)$$

Denote by Δ a triangle in the plane. The main result of the present paper is the following statement.

Theorem. Probability $\mathbf{P}(L(\omega)\subset\Delta)$ for arbitrary triangle has the explicit forms (2)–(8) depending on the value of l.

Proof. Without loss of the generality we assume, that $AB\equiv a$ is the longest side of $\triangle ABC$, $\angle CAB\equiv\alpha$ is the smallest angle, and $\angle ABC\equiv\beta$. Thus, we have $BC=\dfrac{a\sin\alpha}{\sin(\alpha+\beta)}$, $CA=\dfrac{a\sin\beta}{\sin(\alpha+\beta)}$, $\angle BCA=\pi-(\alpha+\beta)$. Since AB is the largest side, then $\angle BCA$ is the biggest angle. Therefore $\alpha\leq\beta\leq\pi-(\alpha+\beta)$.

Covariogram of a triangle Δ with side a has the form (see [3]):

$$C(\Delta,u,l)=\begin{cases}\dfrac{(a\sin\beta-t\sin(u+\beta))^2\sin\alpha}{2\sin\beta\sin(\alpha+\beta)}, & u\in[0,\alpha], t\in[0,\dfrac{a\sin\beta}{\sin(u+\beta)}],\\ \dfrac{(a\sin\alpha\sin\beta-t\sin u\,\sin(\alpha+\beta))^2}{2\sin\alpha\sin\beta\,\sin(\alpha+\beta)}, & u\in[\alpha,\pi-\beta],\\ & t\in[0,\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)\sin u}],\\ \dfrac{(a\sin\alpha-t\sin(u-\alpha))^2\sin\beta}{2\sin\alpha\sin(\alpha+\beta)}, & u\in[\pi-\beta,\pi],\ t\in[0,\dfrac{a\sin\alpha}{\sin(u-\alpha)}],\\ \dfrac{(a\sin\beta+t\sin(u+\beta))^2\sin\alpha}{2\sin\beta\sin(\alpha+\beta)}, & u\in[\pi,\pi+\alpha],\ t\in[0,-\dfrac{a\sin\beta}{\sin(u+\beta)}],\\ \dfrac{(a\sin\alpha\sin\beta+t\sin u\,\sin(\alpha+\beta))^2}{2\sin\alpha\sin\beta\,\sin(\alpha+\beta)}, & u\in[\pi+\alpha,2\pi-\beta],\\ & t\in[0,-\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)\sin u}],\\ \dfrac{(a\sin\alpha+t\sin(u-\alpha))^2\sin\beta}{2\sin\alpha\sin(\alpha+\beta)}, & u\in[2\pi-\beta,2\pi],\ t\in[0,-\dfrac{a\sin\alpha}{\sin(u-\alpha)}].\end{cases}$$

Let consider the following cases

a) $0\leq l\leq\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

Using (1), we get

$$\mathbf{P}(L(\omega)\subset\Delta)=\frac{1}{\pi S(\Delta)+l|\partial\Delta|}\int_0^{\pi}C(\Delta,u,l)\,du=$$

$$\frac{2\sin(\alpha+\beta)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}\times$$

$$\times\left(\int_0^{\alpha}\frac{(a\sin\beta-l\sin(u+\beta))^2\sin\alpha}{2\sin\beta\sin(\alpha+\beta)}du+\right.$$

$$\left.+\int_{\alpha}^{\pi-\beta}\frac{(a\sin\alpha\sin\beta-l\sin u\,\sin(\alpha+\beta))^2}{2\sin\alpha\sin\beta\,\sin(\alpha+\beta)}du+\int_{\pi-\beta}^{\pi}\frac{(a\sin\alpha-l\sin(u-\alpha))^2\sin\beta}{2\sin\alpha\sin(\alpha+\beta)}du\right).$$

We set

$$f_1(x,y) \equiv \frac{\sin\alpha}{\sin\beta}\int_x^y (a\sin\beta - l\sin(u+\beta))^2 du = a^2\sin\alpha\sin\beta(y-x)-$$

$$-4al\sin\alpha\sin\left(\frac{y+x}{2}+\beta\right)\sin\left(\frac{y-x}{2}\right)+\frac{l^2\sin\alpha}{2\sin\beta}((y-x)-\sin(y-x)\cos(y+x+2\beta)),$$

$$f_2(x,y) \equiv \frac{1}{\sin\alpha\sin\beta}\int_x^y (a\sin\alpha\sin\beta - l\sin u\sin(\alpha+\beta))^2 du = a^2\sin\alpha\sin\beta(y-x)-$$

$$4al\sin(\alpha+\beta)\sin\left(\frac{y+x}{2}\right)\sin\left(\frac{y-x}{2}\right)+\frac{l^2\sin^2(\alpha+\beta)}{2\sin\alpha\sin\beta}((y-x)-\sin(y-x)\cos(y+x)),$$

$$f_3(x,y) \equiv \frac{\sin\beta}{\sin\alpha}\int_x^y (a\sin\alpha - l\sin(u-\alpha))^2 du = a^2\sin\alpha\sin\beta(y-x)-$$

$$-4al\sin\beta\sin\left(\frac{y+x}{2}-\alpha\right)\sin\left(\frac{y-x}{2}\right)+\frac{l^2\sin\beta}{2\sin\alpha}((y-x)-\sin(y-x)\cos(y+x-2\alpha)).$$

Hence, for $0 \le l \le \dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ we get

$$\mathbf{P}(L(\omega)\subset\Delta) = \frac{f_1(0,\alpha)+f_2(\alpha,\pi-\beta)+f_3(\pi-\beta,\pi)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}. \tag{2}$$

b) $\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)} \le l \le a\sin\alpha$. We have

$$\mathbf{P}(L(\omega)\subset\Delta) = \frac{f_1(0,\alpha)+f_2(\alpha,\alpha+\varphi_1)+f_2(\pi-\beta-\phi_1,\pi-\beta)+f_3(\pi-\beta,\pi)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}, \tag{3}$$

where $\varphi_1 = \arcsin\dfrac{a\sin\alpha\sin\beta}{l\sin(\alpha+\beta)}-\alpha$, $\phi_1 = \arcsin\dfrac{a\sin\alpha\sin\beta}{l\sin(\alpha+\beta)}-\beta$.

c) $a\sin\alpha \le l \le \min\left\{\dfrac{a\sin\alpha}{\sin(\alpha+\beta)}, a\sin\beta\right\}$, for which we have

$$\mathbf{P}(L(\omega)\subset\Delta) = \frac{1}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}\times$$
$$\times(f_1(0,\alpha)+f_2(\alpha,\alpha+\varphi_1)+ \tag{4}$$
$$+f_2(\pi-\beta-\phi_1,\pi-\beta)+f_3(\pi-\beta,\pi-\beta+\varphi_2)+f_3(\pi-\phi_2,\pi)),$$

where $\varphi_2 = \alpha+\beta-\pi+\arcsin\dfrac{a\sin\alpha}{l}$, $\phi_2 = \arcsin\dfrac{a\sin\alpha}{l}-\alpha$:

c1) if $\sin\beta \le \dfrac{\sin\alpha}{\sin(\alpha+\beta)}$,we consider $a\sin\beta \le l \le \dfrac{a\sin\alpha}{\sin(\alpha+\beta)}$, so

$$\mathbf{P}(L(\omega)\subset\Delta)=\frac{1}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}\times$$
$$\times\left(f_1(0,\varphi_3)+f_1(\alpha-\phi_3,\alpha)+f_2(\alpha,\alpha+\varphi_1)+f_2(\pi-\beta-\phi_1,\pi-\beta)+\right. \tag{5}$$
$$\left.+f_3(\pi-\beta,\pi-\beta+\varphi_2)+f_3(\pi-\phi_2,\pi)\right),$$

where $\varphi_3=\arcsin\dfrac{a\sin\beta}{l}-\beta$, $\phi_3=\alpha+\beta-\pi+\arcsin\dfrac{a\sin\beta}{l}$;

c2) if $\dfrac{\sin\alpha}{\sin(\alpha+\beta)}\le\sin\beta$, we consider $\dfrac{a\sin\alpha}{\sin(\alpha+\beta)}\le l\le a\sin\beta$, then

$$\mathbf{P}(L(\omega)\subset\Delta)=\frac{f_1(0,\alpha)+f_2(\alpha,\alpha+\varphi_1)+f_3(\pi-\phi_2,\pi)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}. \tag{6}$$

d) $\max\left\{\dfrac{a\sin\alpha}{\sin(\alpha+\beta)},a\sin\beta\right\}\le l\le\dfrac{a\sin\beta}{\sin(\alpha+\beta)}$,

$$\mathbf{P}(L(\omega)\subset\Delta)=\frac{f_1(0,\varphi_3)+f_1(\alpha-\phi_3,\alpha)+f_2(\alpha,\alpha+\varphi_1)+f_3(\pi-\phi_2,\pi)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}. \tag{7}$$

e) $\dfrac{a\sin\beta}{\sin(\alpha+\beta)}\le l\le a$ we have

$$\mathbf{P}(L(\omega)\subset\Delta)=\frac{f_1(0,\varphi_3)+f_3(\pi-\phi_2,\pi)}{\pi a^2\sin\alpha\sin\beta+2al(\sin\alpha+\sin\beta+\sin(\alpha+\beta))}. \tag{8}$$

Obviously, if $l>a$, the probability $\mathbf{P}(L(\omega)\subset\Delta)$ is zero. □

Particularly, for regular triangle with a side a and $\alpha=\beta=60^\circ$, among all 5 subcases a)–e) there are only two cases, namely

$$0\le l\le\sin\alpha\quad\text{and}\quad\sin\alpha\le l\le a$$

and result of Theorem coincides with the result of [7] (Eqs. (4.3), (4.4)) for a regular triangle.

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