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M a t h e m a t i c s

ON A VERSION OF FIXED POINT THEOREM

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In the present short note we prove a version of the classical fixed-point theorem, as well as present its application.

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The classical Banach fixed-point theorem, known also as contraction mapping theorem, can be formulated as follow: if (X,ρ) is a complete metric space and

$$A:(X,\rho)\to(X,\rho)$$

is a contraction operator, i.e. there is $0\leq\alpha<1$ such that

$$\rho(Ax_1,Ax_2)\leq\alpha\rho(x_1,x_2),$$

then the operator A has a unique fixed point. Precisely, there exists a unique $x_0\in X$ such that $Ax_0=x_0$.

This theorem, also known as contraction mapping principle, has many applications in functional analysis, as well as in various problems of analysis, especially in proofs of existence and uniqueness of solutions of certain differential equations.

In this note we introduce a new version of this principle for factor spaces. Let X be a Banach space and X_1 be a closed subspace of X. The factor space X/X_1 consists of classes $[x]=\{x+y;\ y\in X_1\}$, generated by elements $x\in X$. It is a Banach space with the norm defined by (see [1])

$$||[x]||_{X/X_1}=\inf\{||x+y||;\ y\in X_1\}.$$

Now let us consider an operator A acting from X to X/X_1. Analogously, we introduce the two definitions.

D e f i n i t i o n 1 . Operator A is called contraction, if there is $0\leq\alpha<1$ such that

$$||Ax_1-Ax_2||_{X/X_1}\leq\alpha||x_1-x_2||_X$$

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for every $x_1, x_2 \in X$.

Definition 2. $x_0 \in X$ is called a fixed-point for the operator A, if $Ax_0 = [x_0]$. Our result is the following version of the fixed point theorem.

Theorem. Let X be a Banach space and X_1 be a closed subspace of X. If $A : X \to X/X_1$ is a contraction mapping, then it has a fixed point.

Proof. Let $\alpha \in [0;1)$ be the number, for which the following inequality holds

$$||Ax_1 - Ax_2||_{X/X_1} \le \alpha ||x_1 - x_2||_X.$$

If $\alpha = 0$, then the statement is obvious.

Now, let $\alpha > 0$ and $x_0 \in X$. If x_0 is not a fixed-point, then we can find $x_1 \in X$ from the class Ax_0 such that

$$||[x_1] - [x_0]||_{X/X_1} + \alpha^0 = \inf\{||Ax_0 - x_0 + y||;\ y \in X_1\} + \alpha^0 \ge ||x_1 - x_0||_X.$$

If x_1 is not a fixed-point, then we can choose x_2 from the class Ax_1 such that

$$||[x_2] - [x_1]||_{X/X_1} + \alpha \ge ||x_2 - x_1||_X.$$

Repeating this iteration, we either find a fixed-point or construct a sequence

$$x_0, x_1, x_2, \dots, x_n, \dots \in X$$

such that x_{n+1} is from the class Ax_n, i.e. $[x_{n+1}] = Ax_n$ and

$$||[x_{n+1}] - [x_n]||_{X/X_1} + \alpha^n \ge ||x_{n+1} - x_n||_X.$$

Now consider the series

$$[x_0] + \sum_0^\infty ([x_{n+1}] - [x_n])$$

in the factor space X/X_1.

Lets show that this series converges in X/X_1 space. Indeed, by mathematical induction we can easily obtain

$$||[x_{n+1}] - [x_n]||_{X/X_1} = ||Ax_n - Ax_{n-1}||_{X/X_1} \le \alpha ||x_n - x_{n-1}||_X \le$$
$$\le \alpha ||[x_n] - [x_{n-1}]||_{X/X_1} + \alpha^{n-1} = \alpha ||[x_n] - [x_{n-1}]||_{X/X_1} + \alpha^n \le$$
$$\le \alpha(\alpha ||[x_{n-1}] - [x_{n-2}]||_{X/X_1} + \alpha^{n-1}) + \alpha^n = \alpha^2 ||[x_{n-1}] - [x_{n-2}]||_{X/X_1} + 2\alpha^n \le$$
$$\le \dots \le \alpha^n ||[x_1] - [x_0]||_{X/X_1} + n\alpha^n$$

and, since α is a positive number less than one, the numerical series

$$\sum_0^\infty \alpha^n (||[x_1] - [x_0]||_{X/X_1} + n)$$

converges. Therefore, the series

$$\sum_0^\infty ||[x_{n+1}] - [x_n]||_{X/X_1}$$

converges too and the sequence

$$[x_n] = [x_0] + \sum_0^n ([x_{k+1}] - [x_k])$$

is convergent in the factor space X/X_1. Denote the limit of $[x_n]$ by $[a]$. Since

$$||x_{n+1}-x_n|| \leq ||[x_{n+1}]-[x_n]||+\alpha^n,$$

where on the right hand side we have elements of two convergent series, the series $x_0+\sum_0^\infty(x_{n+1}-x_n)$ is convergent in X. Thus the sequence $x_n = x_0+\sum_0^n(x_{n+1}-x_n)$ is convergent, and there exists $b \in X$ such that $x_n \to b$. Lets prove that $b \in [a]$. Indeed, we have

$$||[x_n]-[b]||_{X/X_1} \leq ||x_n-b||_X.$$

Therefore, $[x_n] \to [b]$ on the other hand $[x_n] \to [a]$ and so we get $[a] = [b]$. Taking limits in $[x_{n+1}] = Ax_n$ we will obtain $[b] = [a] = Ab$. Thus $[b] = Ab$ and b is a fixed point.

The uniqueness of the fixed-point general is not true for factor spaces.

Consider the Banach space $X = C[0;1]$ of real continuous functions $x(t)$ on $[0;1]$ with usual maximum norm

$$||x(t)|| = \max\{|x(t)|;\ t \in [0;1]\}.$$

Let X_1 be the subspace of functions, which vanish at 0. Obviously it is a closed subspace of X. It can be easily seen that the factor space X/X_1 can be identified with the space of real numbers equipped with usual norm. Indeed, the class $[x(t)]$ generated by function $x(t)$ coincides with class $[x(0)]$ generated by function $x(t) \equiv x(0)$ and

$$||[x(0)]||_{X/X_1} = \inf\{||x(0)+x_1(t)||;\ \ x_1(0)=0\} = |x(0)|.$$

Observe that the operator $A: X \to X/X_1$ acting by equation

$$Ax = \left[\frac{1}{2}\int_0^1 x(t)dt\right]$$

is a linear contraction.

On the other hand, in any class $[x]$ we can find a fixed-point $y \in [x]$, by taking

$$y(t) = x(0)+2x(0)t.$$

Indeed, $y(0) = x(0)$, so $y \in [x]$ and $Ay = [x(0)] = [y]$.

Now let us present one application of the theorem.

Let $L(X,Y)$ be the space of linear bounded operators from a Banach space X to a Banach space Y, and let $L_1(X,Y)$ be the subset of surjective operators, i.e. operators $A \in L(X,Y)$ such that $Y = A(X)$.

Corollary. $L_1(X,Y)$ is an open subset of $L(X,Y)$ (see [2]).

Proof. Let us show that if $A \in L_1(X,Y)$ and $B \in L(X,Y)$, then $A+B \in L_1(X,Y)$ provided the norm of B is small enough. This will prove the Corollary. If $\ker A = \{x;\ Ax = 0\}$, then we can define the operator

$$A_1 : X/\ker A \to Y$$

as $A_1[x] = Ax$. Since $A: X \to Y$ is surjective, $A_1 : X/\ker A \to Y$ will be bijective. By Banach's theorem there exists an inverse bounded operator

$$A_1^{-1} : Y \to X/\ker A.$$

Now for any $y \in Y$ let us consider the operator $C : X \to X/\ker A$ defined by

$$Cx = A_1^{-1}y - A_1^{-1}Bx,$$

where $B \in L(X,Y)$ is arbitrary so far. By choosing the norm of $||B|| < ||A_1^{-1}||^{-1}$, we obtain that C is a contraction:

$$||Cx_1 - Cx_2||_{X/\ker A} \leq \alpha ||x_1 - x_2||_X,$$

where

$$\alpha = ||A_1^{-1}B|| \leq ||A_1^{-1}|| \; ||B|| < 1.$$

Applying Theorem we obtain that C has a fixed-point x, i.e.

$$[x] = A_1^{-1}y - A_1^{-1}Bx,$$

and applying the operator A_1 on both sides, we obtain

$$Ax = A_1[x] = y - Bx$$

or

$$(A+B)x = y.$$

We have shown that if $||B|| < ||A_1^{-1}||^{-1}$, then $A+B$ is surjective.

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R E F E R E N C E S

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