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M a t h e m a t i c s

ON THE IDENTIFICATION OF THE SOURCE OF EMISSION ON THE PLANE

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We consider the problem of identification of the position and the moment of the beginning of a radioactive source emission on the plane. The acts of emission constitute inhomogeneous Poisson processes and are registered by K detectors on the plane. We suppose that the moments of arriving of the signals at the detectors are measured with some small errors. Then, using these estimate, we construct the estimators of the position of source and the moment of the beginning of emission. We study the asymptotic properties of these estimators for large signals and prove their consistency.

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Introduction. There are given K detectors $D_1,\dots,D_K$ placed at the points $\vartheta_k = (x_k, y_k), k = 1,\dots,K$, on the plane. We suppose that at an unknown location $\vartheta_0 = (x_0, y_0) \in \Theta \subset \mathcal{R}^2$ at an unknown moment τ_0 a radioactive device starts emission. The detectors receive signals, and based on these detections, the statistician has to estimate the position of the source and the time τ_0 of the beginning of emission. We obtain a similar mathematical model of observations in the case of a weak optical source emitting photons. Note that in the problem of GPS-localization we have the same mathematical model for the inverse experiment. We have K emitters of signals $D_1,\dots,D_K$ received by the device D_0, and using the observations of these signals, it is necessary to estimate the position of the device.

An example of such a model of observations is given in the Fig. 1, where D_0 is the position of the source and $D_1 - D_5$ are the detectors.

Due to importance of this problem in many applications there exists a large amount of literature on the identification of radioactive sources of engineering level (see [1–4]). To the best of our knowledge, the mathematical study of such problems

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is not yet sufficiently developed. This work is the continuation of the study started in papers [5–7].

We consider the case when the signals received by the detectors are inhomogeneous Poisson processes $X^K = (X_1, \ldots, X_K)$, where the process $X_k = (X_k(t), 0 \le t \le T)$ has intensity function

$$\lambda(\vartheta, t) = nS_k(t - \tau_k(\vartheta_0))\,\bar\psi(t - \tau_k(\vartheta_0)) + n\lambda_0, \quad 0 \le t \le T. \tag{1}$$

Here $nS(\cdot)$ is a known positive continuous function (intensity of the signal), $\tau_k(\vartheta_0)$ is the moment of arriving of the signal at the k-th detector and $n\lambda_0$ is the background noise. The time of arriving $\tau_k = \tau_k(\vartheta_0)$ can be written as follows

$$\tau_k = \tau_0 + \nu^{-1}\|\vartheta_k - \vartheta_0\| = \tau_0 + \tau_{k,0}, \tag{2}$$

where $\tau_0 \in \mathcal{T}$ is the moment of beginning of emission, $\nu > 0$ is the rate of propagation of the signals, $\|\cdot\|$ is the Euclidean norm in $\mathcal{R}^2$ and $\|\vartheta_k - \vartheta_0\|$ is the distance between the point of emission and the k-th detector, $\tau_{k,0}$ is the time to reach of the k-th detector after the beginning of emission [8].

The function $\bar\psi(t) = 0$ for $t < 0$ reflects the form of the signal at the moment of its arriving. We consider three different cases: smooth $\psi_\delta(\cdot)$, cusp-type $\psi_{\delta,\kappa}$ and change-point type $\psi(\cdot)$, where

$$\psi_\delta(t) = \frac{t}{\delta}\,\mathbb{I}_{\{0 \le t \le \delta\}} + \mathbb{I}_{\{t > \delta\}}, \quad \psi(t) = \mathbb{I}_{\{t > 0\}},$$
$$\psi_{\delta,\kappa}(t) = \frac{1}{2}\left(1 + \operatorname{sgn}(2t - \delta)\left|\frac{2t}{\delta} - 1\right|^{\kappa}\right)\mathbb{I}_{\{0 \le t \le \delta\}} + \mathbb{I}_{\{t > \delta\}},$$

The parameter $\delta > 0$ is known and small. In the cusp case $\kappa \in \left(0, \frac{1}{2}\right)$. The examples of such functions are given in the Fig. 1. The case b) in Fig. 1 corresponds to the function $\psi_{\delta,\kappa}(\cdot)$ with the value $\kappa = \frac{1}{2}$ and the case e) is obtained if in $\psi_{\delta,\kappa}(\cdot)$ the parameter $\kappa \in (-1, 0)$.

Consider the problem of estimation of the position $\vartheta_0 = (x_0, y_0)$ and the moment τ_* by the observations $X^K = (X_1, \ldots, K)$. The estimators of these quantities are studied in the asymptotics of large signals, i.e. as $n \to \infty$.

Recall that the cases a)–d) with $\tau_0 = 0$, i.e. a known moment of beginning of emission, were considered in [5–7]. It was shown that the Bayes estimators $\tilde\vartheta_n$ of the parameter ϑ_0 have the following limits

$$\text{a)} \quad \sqrt{n}\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_1, \qquad \text{b)} \quad \sqrt{n \ln n}\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_2,$$
$$\text{c)} \quad n^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_3, \qquad \text{d)} \quad n\left(\tilde\vartheta_n - \vartheta_0\right) \Longrightarrow \zeta_4,$$

where ζ_i, $i = 1, \ldots, 4$, are some random vectors all having polynomial moments (for details, see [5–7]).

It is possible to consider a different statement of the problem. Let us study K independent Poisson processes $X_1, \ldots, X_K$ with intensity functions (1) and estimate the parameters $\tau_1, \ldots, \tau_K$. The corresponding MLE $\hat\tau_{k,n}$, $k = 1, \ldots, K$, and BE

$\tilde{\tau}_{k,n},\ k=1,\dots,K$, have the similar properties. For example,

$$\text{a)}\quad \sqrt{n}\left(\hat{\tau}_{k,n}-\tau_k\right)\Longrightarrow \xi_{k,1},\qquad \text{b)}\quad \sqrt{n\ln n}\left(\hat{\tau}_{k,n}-\tau_k\right)\Longrightarrow \xi_{k,2}, \tag{3}$$

$$\text{c)}\quad n^{\frac{1}{2\kappa+1}}\left(\hat{\tau}_{k,n}-\tau_k\right)\Longrightarrow \xi_{k,3},\qquad \text{d)}\quad n\left(\hat{\tau}_{k,n}-\tau_k\right)\Longrightarrow \xi_{k,4}, \tag{4}$$

$$\text{e)}\quad n^{\frac{1}{\kappa+1}}\left(\hat{\tau}_{k,n}-\tau_k\right)\Longrightarrow \xi_{k,5}. \tag{5}$$

For the cases a) and d) see [9], cases c) and e) (for $\kappa\in(-1,0)$) were studied in [10, 11], respectively. The case b) follows from the results presented in [5].

In this work we consider the estimation of parameters τ_0,ϑ_0 in two steps. First, we estimate K moments $\tau_1,\dots,\tau_K$. Then having these estimators with properties (3)–(5) we estimate τ_0,ϑ_0.

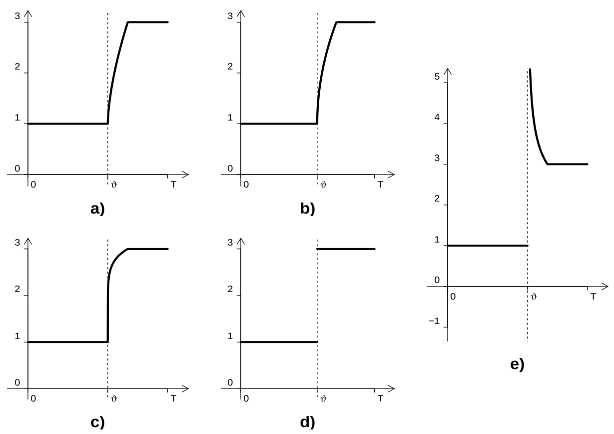

Fig. 1. Examples of functions $\bar{\psi}(\cdot)$. Here a) and b) smooth cases; c) cusp-type case; d) change-point case; e) non bounded intensity case.

Geometrical View. Consider the problem of estimation of location ϑ_0 and the moment of emission τ_0 in the situation, where there is no errors, i.e. the detectors measure $\tau_k=\tau_k(\vartheta_0)$ exactly. If we know τ_0, then 3 detector are enough to find the location of the source (see, e.g., [5]). Now we want to see geometrically whether in the case of unknown τ_0 3 detectors are sufficient to find ϑ_0. Suppose that we have 2 detectors D_1 and D_2 and we know exactly moments τ_1 and τ_2 when signals had arrived. We denote $r_1=\|\vartheta_1-\vartheta_0\|$ the distance between our device and the detector D_1, in the same way r_2 is the distance between device and the detector D_2. r_1 and r_2 are unknown, but we can calculate their difference. If $r_1=\nu(\tau_1-\tau_0)$ and $r_2=\nu(\tau_1-\tau_0)$, then we have $r_1-r_2=\nu(\tau_1-\tau_2)$.

We denote $r=\nu(\tau_1-\tau_2)$, so the difference of distances is r. If we look all possible locations of the device it is a hyperbola branch with focuses D_1 and D_2. For every point (x,y) on this hyperbola we have $(x-x_1)^2+(y-y_1)^2=r_1^2, (x-x_2)^2+$ $+(y-y_2)^2=r_2^2$ and $r_2^2=(r_1-r)^2$, from this three equations we obtain the equation of our hyperbola:

$$(x_1-x_2)(2x-x_1-x_2)+(y_1-y_2)(2y-y_1-y_2)-r^2-2r\sqrt{(x-x_1)^2+(y-y_1)^2}=0.$$

Now if we add the third detector D_3, we can construct another hyperbola branch corresponding to the focuses D_3 and D_2 with the equation:

$$(x_2-x_3)(2x-x_2-x_3)+(y_2-y_3)(2y-y_2-y_3)-r'^2-2r'\sqrt{(x-x_2)^2+(y-y_2)^2}=0,$$

where $r'=\nu(\tau_2-\tau_3)$. The device will be at the point of intersection of these two hyperbolas. We can see in Fig. 2 two points of intersection of hyperbolas. Thus, in general we can not identify the location of device with 3 detectors.

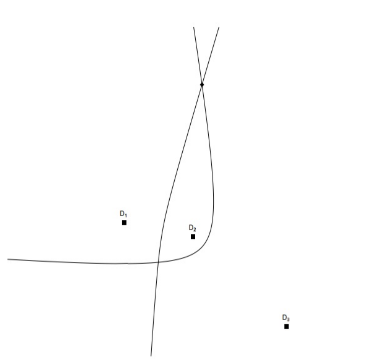

Fig. 2. Two hyperbolas with focuses $D_1,D_2; D_2,D_3$.

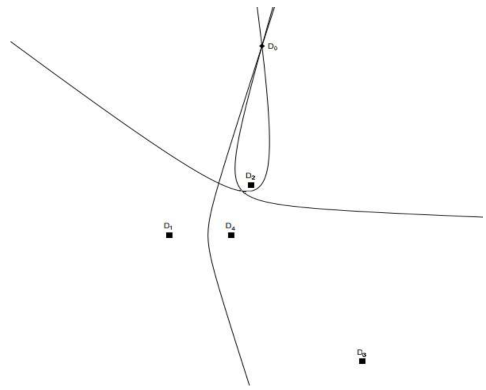

Fig. 3. Detection with four detectors.

Now we will try to find out whether four detectors are enough to identify the exact location of the device. So we consider the signals of D_1-D_3 detectors that show us two possible points in the plane. Hence we have 2 intersection points of hyperbolas denoted by P_1 and P_2. So we want to find a position D_4 such that the hyperbola branch with focuses in D_4 and at the location of one of the other detectors passes through only one of the points P_1 and P_2. To this end we will find all possible focuses of hyperbola branch that passes by points P_1 and P_2. Hence, if F_1 and F_2 are those focuses, we have

$$\rho(P_1,F_1)-\rho(P_1,F_2)=\rho(P_2,F_1)-\rho(P_2,F_2)$$

or

$$\rho(P_1,F_1)-\rho(P_1,F_2)=-\rho(P_2,F_1)-\rho(P_2,F_2).$$

So we have that $\rho(P_1,F_1)-\rho(P_2,F_1)=\rho(P_1,F_2)-\rho(P_2,F_2)$ (or similar for the second equation), which means that all focuses of hyperbola branch passing by P_1 and P_2 are located on the other hyperbola branch with the focuses F_1 and F_2. Thus to identify the location of device we need at least four detectors.

Main Results. We have K independent Poisson processes $X^K=(X_1,\ldots,X_K)$, where the random process $X_k=(X_k(t),\ 0\le t\le T)$ has intensity function (1) and we have to estimate the parameters τ_0,ϑ_0 by observations X^K. We will this problem solve in two steps. First we obtain K independent estimators $\bar\tau_{k,n},\ k=1,\ldots,K$, of the moments of signals arriving at the detectors. Then having these estimators we consider the problem of estimation of τ_0,ϑ_0. The advantage of this approach is its computational simplicity with respect to the traditional maximum likelihood approach. Recall that in maximum likelihood approach all data have to come to one center of simultaneous treatment and then the estimators are obtained as a result of maximization of the likelihood ratio function of three variables

$$\ln L\left(\tau,\vartheta,X^K\right)=\sum_{k=1}^{K}\int_{\tau_k(\vartheta)}^{T}\ln\left(1+\frac{S_k\left(t-\tau_k\left(\vartheta\right)\right)\psi_\delta\left(t-\tau_k\left(\vartheta\right)\right)}{\lambda_0}\right)dX_k\left(t\right)-$$
$$-n\sum_{k=1}^{K}\int_{\tau_k(\vartheta)}^{T}S_k\left(t-\tau_k\left(\vartheta\right)\right)\psi_\delta\left(t-\tau_k\left(\vartheta\right)\right)dt,$$

where $\tau_k\left(\vartheta\right)=\tau_0+\nu^{-1}\left\|\vartheta_k-\vartheta\right\|$.

In our approach the estimators $\hat\tau_{k,n}$ can be calculated in each detector and then transmitted to the center, where the problem of estimation is reduced to the solution of linear equations.

The convergences (3)–(5) can be summarized in the following representations

$$\hat\tau_{k,n}=\tau_0+\tau_{k,0}+\varphi_n\eta_{k,n},\quad \varphi_n\longrightarrow 0, \tag{6}$$

where $\eta_{k,n}$ converge in distribution to the corresponding random variable ξ_k.

The unknown parameters satisfy the following equations

$$\nu^2\left(\tau_k-\tau_0\right)^2=\left(x_k-x_0\right)^2+\left(y_k-y_0\right)^2,\quad k=1,\ldots,K.$$

We have $\quad \nu^2\tau_k^2=x_k^2+y_k^2+x_0^2+y_0^2-\nu^2\tau_0^2-2x_kx_0-2y_ky_0+2\nu^2\tau_k\tau_0.$

Let us denote

$$\gamma_1=x_0,\quad \gamma_2=y_0,\quad \gamma_3=\tau_0,\quad \gamma_4=\frac{1}{2}\left(x_0^2+y_0^2-\nu^2\tau_0^2\right),$$
$$\gamma=\left(\gamma_1,\gamma_2,\gamma_3,\gamma_4\right),\quad z_k=\frac{1}{2}\left(x_k^2+y_k^2-\nu^2\tau_k^2\right).$$

Then this equation can be written as follows

$$x_k\gamma_1+y_k\gamma_2-\nu^2\tau_k\gamma_3+\gamma_4=z_k,\quad k=1,\ldots,K. \tag{7}$$

We have the "observations" $\hat\tau_{k,n}$ and $z_{k,n}=\frac{1}{2}\left(x_k^2+y_k^2-\nu^2\hat\tau_{k,n}^2\right)$. Therefore, we define the estimator $\gamma_n^*=\left(\gamma_{1,n}^*,\gamma_{2,n}^*,\gamma_{3,n}^*,\gamma_{4,n}^*\right)$ using the least squares approach as follows

$$\gamma_n^*=\arg\min_{\gamma}S_n\left(\gamma\right),\quad S_n\left(\gamma\right)=\sum_{k=1}^{K}\left[z_{k,n}-x_k\gamma_1-y_k\gamma_2+\nu^2\hat\tau_{k,n}\gamma_3-\gamma_4\right]^2.$$

It will be convenient to denote $\nu^2\hat\tau_{k,n}=-\hat\rho_{k,n}$. Therefore, the least squares estimator γ_n^* is the solution of the equations

$$\frac{\partial S_n\left(\gamma\right)}{\partial\gamma_l}=0,\quad l=1,\ldots,4,$$

which can be written as

$$\sum_{k=1}^{K}x_k^2\gamma_{1,n}^*+\sum_{k=1}^{K}x_ky_k\gamma_{2,n}^*+\sum_{k=1}^{K}x_k\hat\rho_{k,n}\gamma_{3,n}^*+\sum_{k=1}^{K}x_k\gamma_{4,n}^*=\sum_{k=1}^{K}x_kz_{k,n},$$
$$\sum_{k=1}^{K}x_ky_k\gamma_{1,n}^*+\sum_{k=1}^{K}y_k^2\gamma_{2,n}^*+\sum_{k=1}^{K}y_k\hat\rho_{k,n}\gamma_{3,n}^*+\sum_{k=1}^{K}y_k\gamma_{4,n}^*=\sum_{k=1}^{K}y_kz_{k,n},$$
$$\sum_{k=1}^{K}x_k\hat\rho_{k,n}\gamma_{1,n}^*+\sum_{k=1}^{K}\hat\rho_{k,n}y_k\gamma_{2,n}^*+\sum_{k=1}^{K}\hat\rho_{k,n}^2\gamma_{3,n}^*+\sum_{k=1}^{K}\hat\rho_{k,n}\gamma_{4,n}^*=\sum_{k=1}^{K}\hat\rho_{k,n}z_{k,n},$$
$$\sum_{k=1}^{K}x_k\gamma_{1,n}^*+\sum_{k=1}^{K}y_k\gamma_{2,n}^*+\sum_{k=1}^{K}\hat\rho_{k,n}\gamma_{3,n}^*+K\gamma_{4,n}^*=\sum_{k=1}^{K}z_{k,n},$$

or in matrix form

$$\mathbb{A}_n\gamma_n^* = Z_n, \quad \gamma_n^* = \mathbb{A}_n^{-1}Z_n$$

with obvious notations. Since $\hat\tau_{k,n} = \tau_0 + \tau_{k,0} + \varphi_n\eta_n \longrightarrow \tau_k = \tau_0 + \tau_{k,0}$, the matrix $\mathbb{A}_n$ converges in probability $\mathbb{A}_n = \mathbb{A}_0 + \varphi_n B_n \longrightarrow \mathbb{A}_0$, where the matrix

$$\mathbb{A}_0 = \begin{pmatrix} \|x\|_K^2, & \langle x,y\rangle_K, & +\langle x,\rho\rangle_K, & \langle x,1\rangle_K \\ \langle x,y\rangle_K, & \|y\|_K^2, & +\langle y,\rho\rangle_K, & \langle y,1\rangle_K \\ \langle x,\rho\rangle_K, & \langle y,\rho\rangle_K, & +\|\rho\|^2, & \langle \rho,1\rangle_K \\ \langle x,1\rangle_K, & \langle y,1\rangle_K, & +\langle 1,\rho\rangle_K, & K \end{pmatrix}.$$

Here $\rho = \left(-\nu^2\tau_1, \ldots, -\nu^2\tau_K\right)$ and

$$\|a\|_K^2 = \sum_{k=1}^K a_k^2, \qquad \langle a,b\rangle_K = \sum_{k=1}^K a_k b_k.$$

Further we have convergence in probability

$$Z_{1,n} = \langle x,z_n\rangle_K \longrightarrow \langle x,z\rangle_K = Z_1, \qquad Z_{2,n} = \langle y,z_n\rangle_K \longrightarrow \langle y,z\rangle_K = Z_2,$$
$$Z_{3,n} = \langle \hat\tau_n,z_n\rangle_K \longrightarrow \langle \tau,z\rangle_K = Z_3, \qquad Z_{4,n} = \langle 1,z_n\rangle_K \longrightarrow \langle 1,z\rangle_K = Z_4,$$

or $Z_n \to Z$, where the vector $Z = (Z_1, \ldots, Z_4)$. Then we can write (7) as

$$\mathbb{A}_n\gamma_n^* = Z_n \quad \text{and} \quad \gamma_n^* = \mathbb{A}_n^{-1}Z_n.$$

We study the asymptotic ($n \to \infty$) behavior of the estimator γ_n^*.

Conditions $\mathscr{C}$.

1. The set $\Theta \subset \mathcal{R}^2$ is open, convex and bounded.

2. The set $\mathcal{T} = (T_i, T_f)$ is such that $\tau_k \in (0,T)$ for all $\vartheta_0 \in \Theta$.

3. The estimators $\hat\tau_{k,n}$, $k = 1, \ldots, K$, admit the representation (6), where $\varphi_n \to 0$ and the random variables $\eta_{k,n}$, $k = 1, \ldots, K$, are bounded in probability.

4. There are at least four detectors, which are not on the same line and the matrix $\mathbb{A}_0$ is non degenerate

$$\inf_{T_i \le \tau_0 \le T_f} \inf_{\|e\|_4 = 1} e^\top \mathbb{A}_0\, e > 0,$$

where $e \in \mathcal{R}^4$.

Note that all these conditions in the case of known τ_0 are fulfilled in the problems considered in the works [5–7].

Therefore we proved the following result.

Theorem . *Let conditions $\mathscr{C}$ be satisfied, then estimator γ_n^* is consistent.*

It can be verified that since the matrix $\mathbb{A}_0$ is uniformly non-degenerate, we have

$$\mathbb{A}_n \longrightarrow \mathbb{A}_0, \quad Z_n \to Z, \quad \mathbb{A}_n^{-1} \longrightarrow \mathbb{A}_0^{-1}, \quad \gamma_n^* \longrightarrow \mathbb{A}_0^{-1}Z = \gamma,$$

where $\gamma = \left(x_0, y_0, \tau_0, 2^{-1}\left(x_0^2 + y_0^2 - \nu^2\tau_0^2\right)\right)$.

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ՀԱՐԹՈՒԹՅԱՆ ՎՐԱ ՃԱՌԱԳԱՅԹՄԱՆ ԱՂԲՅՈՒՐԻ ՀԱՅՏՆԱԲԵՐՈՒՄԸ

Աշխատանքում դիտարկվում է հարթության վրա ռադիոակտիվ աղբյուրի դիրքի և ճառագայթման սկսվելու պահի նույնականացման խնդիրը: Ճառագայթման ակտերը ոչ համասեռ պուասոնյան գործընթացներ են՝ գրանցված K դետեկտորների միջոցով: Մենք համարում ենք, որ ազդանշանները հասնելու պահերը որոշվում են փոքր սխալներով: Այնուհետև օգտագործելով այս չափումները մենք կառուցում ենք աղբյուրի դիրքի և ճառագայթման մեկնարկի պահի գնահատականներ: Մենք ուսումնասիրում ենք այդ գնահատականների հատկությունները մեծ ազդանշանների սահմանում և ցույց ենք տալիս դրանց հետևողականությունը: