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Mathematics

ON THE DIMENSION OF SPACES OF ALGEBRAIC CURVES PASSING THROUGH n-INDEPENDENT NODES

H. A. HAKOPIAN *, H. M. KLOYAN **

Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia

Let the set of nodes $\mathcal{X}$ in the plain be n-independent, i.e., each node has a fundamental polynomial of degree n. Suppose also that $|\mathcal{X}| = (n+1)+n+\cdots+$ $+(n-k+4)+2$ and $3 \le k \le n-1$. We prove that there can be at most 4 linearly independent curves of degree less than or equal to k passing through all the nodes of $\mathcal{X}$. We provide a characterization of the case when there are exactly 4 such curves. Namely, we prove that then the set $\mathcal{X}$ has a very special construction: all its nodes but two belong to a (maximal) curve of degree $k-2$. At the end, an important application to the Gasca-Maeztu conjecture is provided.

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Introduction. Denote the space of all bivariate polynomials of total degree $\le n$ by Π_n, i.e., $\Pi_n = \left\{\sum_{i+j\le n} a_{ij}x^iy^j\right\}$. We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Consider a set of s distinct nodes $\mathcal{X} = \mathcal{X}_s = \{(x_1,y_1),(x_2,y_2),\ldots,(x_s,y_s)\}$. The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions

$$p(x_i,y_i) = c_i, \quad i = 1,\ldots,s, \tag{1}$$

is called interpolation problem.

A polynomial $p \in \Pi_n$ is called a fundamental polynomial for a node $A \in \mathcal{X}$ if $p(A) = 1$ and $p|_{\mathcal{X}\setminus\{A\}} = 0$, where $p|_{\mathcal{X}}$ means the restriction of p on $\mathcal{X}$. We denote the fundamental polynomial by $p^{\star}_A$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of $\mathcal{X}$ but one, since it is a nonzero constant times a fundamental polynomial.

Definition 1. *The interpolation problem with a set of nodes $\mathcal{X}_s$ and Π_n is called n-poised if for any data $(c_1,\ldots,c_s)$ there is a unique polynomial $p \in \Pi_n$ satisfying the interpolation conditions* (1).

* E-mail: hakop@ysu.am ** E-mail: arutkloyan@gmail.com

A necessary condition of poisedness is $|\mathcal{X}_s| = s = N$.

P r o p o s i t i o n 1. *A set of nodes $\mathcal{X}_N$ is n-poised if and only if*

$$p \in \Pi_n \quad and \quad p|_{\mathcal{X}_N} = 0 \quad \Longrightarrow \quad p = 0.$$

Next, let us consider the concept of n-independence (see [1,2]).

D e f i n i t i o n 2. *A set of nodes $\mathcal{X}$ is called n-independent, if all its nodes have n-fundamental polynomials. Otherwise, it is called n-dependent.*

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence of $\mathcal{X}_s$ is $s \le N$.

Some Properties of n-Independent Nodes. Let us start with the following simple (see Lemma 2.2 [3])

L e m m a 1. *Suppose that a node set $\mathcal{X}$ is n-independent and a node $A \notin \mathcal{X}$ has n-fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then the latter node set is n-independent too.*

Denote the distance between the points A and B by $\rho(A,B)$. Let us recall the following (see [4,5])

L e m m a 2. *Suppose that $\mathcal{X}_s = \{A_i\}_{i=1}^s$ is an n-independent set. Then there is a number $\varepsilon > 0$ such that any set $\mathcal{X}'_s = \{A'_i\}_{i=1}^s$, with the property that $\rho(A_i, A'_i) < \varepsilon$, $i = 1, \ldots, s$, is n-independent too.*

Next result concerns the extension of n-independent sets (see Lemma 2.1 [2]).

L e m m a 3. *Any n-independent set $\mathcal{X}$ with $|\mathcal{X}| < N$ can be enlarged to an n-poised set.*

In the sequel we will need the following modification of the above result.

L e m m a 4. *Given n-independent sets $\mathcal{X}_{s_i}$, $i = 1, \ldots, m$, where $|\mathcal{X}_{s_i}| = s_i < N$, a node A and any number $\varepsilon > 0$. Then there is a node A' such that $\rho(A, A') < \varepsilon$ and each set $\mathcal{X}_{s_i} \cup \{A'\}$, $i = 1, \ldots, m$, is n-independent.*

P r o o f. Let us use induction with respect to the number of sets: m. Suppose that we have one set $\mathcal{X}_s$. Since $s < N$, there is a nonzero polynomial $p \in \Pi_n$ such that $p|_{\mathcal{X}_s} = 0$. Now evidently there is a node $B \notin \mathcal{X}$ such that $\rho(A,B) < \varepsilon$ and $p(B) \neq 0$. Thus p is an n-fundamental polynomial of the node B with respect to the set $\mathcal{X} \cup \{B\}$. Hence, in view of Lemma 1, the set $\mathcal{X}_s \cup \{B\}$ is n-independent. Then, assume that Lemma is true in the case of $m-1$ sets, i.e. there is a node B such that $\rho(A,B) < (1/2)\varepsilon$ and each set $\mathcal{X}_{s_i} \cup \{B\}$, $i = 1, \ldots, m-1$, is n-independent. In view of Lemma 2, there is a number $\varepsilon' < (1/2)\varepsilon$ such that for any C with $\rho(C,B) < \varepsilon'$ each set $\mathcal{X}_{s_i} \cup \{C\}$, $i = 1, \ldots, m-1$, is n-independent. Next, in view of first step of induction there is a node A' such that $\rho(A', B) < (1/2)\varepsilon$ and the set $\mathcal{X}_{s_m} \cup \{A'\}$ is n-independent. Now, it is easily seen that A' is a desirable node. □

Denote the linear space of polynomials of total degree at most n vanishing on $\mathcal{X}$ by

$$\mathcal{P}_{n,\mathcal{X}} = \{p \in \Pi_n : p|_{\mathcal{X}} = 0\}.$$

The following two propositions are well-known [2].

P r o p o s i t i o n 2. *For any node set $\mathcal{X}$ we have that*

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - |\mathcal{Y}|,$$

where $\mathcal{Y}$ is a maximal n-independent subset of $\mathcal{X}$.

Proposition 3. *If a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ, then we have that $p = \ell r$, where $r \in \Pi_{n-1}$.*

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter p, say, to denote the polynomial p of degree ≥ 1 and the curve given by the equation $p(x,y) = 0$.

Set $d(n,k) := N_n - N_{n-k} = (1/2)k(2n+3-k)$. The following is a generalization of Proposition 3 (see Prop. 3.1 [6]).

Proposition 4. *Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold:*

$i)$ any subset of q containing more than $d(n,k)$ nodes is n-dependent;

$ii)$ any subset $\mathcal{X}_d$ of q containing exactly $d = d(n,k)$ nodes is n-independent if and only if the following condition holds:

$$p \in \Pi_n \quad and \quad p|_{\mathcal{X}_d} = 0 \Longrightarrow p = qr, \text{ where } r \in \Pi_{n-k}. \tag{2}$$

Thus, according to Proposition 4, $i)$, at most $d(n,k)$ nodes of $\mathcal{X}$ can lie in the curve q of degree $k \leq n$. This motivates the following definition (see Def. 3.1 [6]).

Definition 3. *Given an n-independent set of nodes $\mathcal{X}_s$ with $s \geq d(n,k)$. A curve of degree $k \leq n$ passing through $d(n,k)$ points of $\mathcal{X}_s$ is called maximal.*

We say that a node A of an n-poised set $\mathcal{X}$ *uses* a line ℓ, if the latter divides the fundamental polynomial of A, i.e., $p_A^\star = \ell q$ for some $q \in \Pi_{n-1}$.

Let us bring a characterization of maximal curves (see Prop. 3.3 [6]):

Proposition 5. *Let a node set $\mathcal{X}$ be n-poised. Then a curve μ of degree k, $k \leq n$, is a maximal curve if and only if it is used by any node in $\mathcal{X} \setminus \mu$.*

Next result concerns maximal independent sets in curves (see Prop. 3.5 [5]).

Proposition 6. *Assume that σ is an algebraic curve of degree k without multiple components and $\mathcal{X}_s \subset \sigma$ is any n-independent node set of cardinality s, $s < d(n,k)$. Then the set $\mathcal{X}_s$ can be extended to a maximal n-independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d = d(n,k)$.*

Finally, let us bring a well-known

Lemma 5. *Suppose that m linearly independent curves pass through all the nodes of $\mathcal{X}$. Then for any node $A \notin \mathcal{X}$ there are $m-1$ linearly independent curves in the linear span of given curves, passing through A and all the nodes of $\mathcal{X}$.*

Main Result. Let us start with (see Theorem 1 [7]).

Theorem 1. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-1)+2$ nodes lying in a curve of degree k with $k \leq n$. Then the curve is determined uniquely by these nodes.*

Next result in this series is the following (see Theorem 4.2 [5])

Theorem 2. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-1)+1$ nodes with $k \leq n-1$. Then two different curves of degree k pass through all the nodes of $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but one lie in a maximal curve of degree $k-1$.*

Now let us present the main result of this paper:

Theorem 3. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-2)+2$ nodes with $k \leq n-1$. Then four linearly independent curves of degree less than or equal* *to k pass through all the nodes of $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but two lie in a maximal curve of degree $k-2$.*

Let us mention that the inverse implication here is evident. Indeed, assume that $d(n,k-2)$ nodes of $\mathcal{X}$ are located in a curve μ of degree $k-2$. Therefore, the curve μ is maximal and the remaining two nodes of $\mathcal{X}$, denoted by A and B, are outside of it: $A,B \notin \mu$. Hence we have that

$$\mathcal{P}_{k,\mathcal{X}} = \{p : p \in \Pi_k, p(A) = p(B) = 0\} = \{q\mu : q \in \Pi_2, q(A) = q(B) = 0\}.$$

Thus we readily get that $\dim \mathcal{P}_{k,\mathcal{X}} = \dim \{q \in \Pi_2 : q(A) = q(B) = 0\} = \dim \mathcal{P}_{2,\{A,B\}} = = 6-2 = 4$. In the last equality we use the fact that any two nodes are 2-independent.

We get also that there can be at most 4 linearly independent curves of degree $\leq k$ passing through all the nodes of $\mathcal{X}$.

Before starting the proof of Theorem 3 let us present two lemmas.

Lemma 6. *Assume that $\mathcal{X}$ is an n-independent node set and a node $A \in \mathcal{X}$ has an n-fundamental polynomial $p_A^\star$ such that $p_A^\star(A') \neq 0$. Then we can replace the node A with A' such that the resulted set $\mathcal{X}' := \mathcal{X} \cup \{A'\} \setminus \{A\}$ is again an n-independent. In particular, such replacement can be done in the following two cases:*

i) if a node $A \in \mathcal{X}$ belongs to several components of σ, then we can replace it with a node A', which belongs only to one component of σ;

ii) if a curve q is not a component of an n-fundamental polynomial $p_A^\star$ then we can replace the node A with a node A' lying in q.

Proof. Indeed, notice that $p_A^\star(A') \neq 0$ means that $p_A^\star$ is a fundamental polynomial for the node A' with respect to the set $\mathcal{X}'$. Next, for i) note that a fundamental polynomial of a node A differs from 0 in a neighborhood of A. Finally, for ii) note that q is not a component of $p_A^\star$ means, that there is a point $A' \in q$ such that $p_A^\star(A') \neq 0$. □

Lemma 7. *Assume that the hypotheses of Theorem 3 hold and assume additionally that there is a curve $q_{k-1} \in \Pi_{k-1}$ passing through all the nodes of $\mathcal{X}$. Then all the nodes of $\mathcal{X}$ but two lie in a maximal curve μ of degree $k-2$.*

Proof. First note that the curve q_{k-1} is of exact degree $k-1$, since it passes through more than $d(n,k-2)$ n-independent nodes. This implies also that q_{k-1} has no multiple component. Therefore, in view of Proposition 6, we can extend the set $\mathcal{X}$ till a maximal n-independent set $\mathcal{Y} \subset q_{k-1}$, by adding $n-k+1$ nodes, i.e.,

$$\mathcal{Y} = \mathcal{X} \cup \mathcal{A}, \text{ where } \mathcal{A} = \{A_0, \ldots, A_{n-k}\}.$$

In view of Lemma 6, i), we may suppose that the nodes from $\mathcal{A}$ are not intersection points of the components of the curve q_{k-1}.

Next, we are going to prove that these $n-k+1$ nodes are collinear together with $m \geq 2$ nodes from $\mathcal{X}$. To this end denote the line through the nodes A_0 and A_1 by ℓ_{01}. Then for each $i = 2\ldots, n-k$ choose a line ℓ_i passing through the node A_i, which is not a component of q_{k-1}. We require also that each line passes through only one of the mentioned nodes and therefore the lines are distinct.

Now suppose that $p \in \Pi_k$ vanishes on $\mathcal{X}$. Consider the polynomial $r = p\ell_{01}\ell_2 \cdots \ell_{n-k}$. We have that $r \in \Pi_n$ and r vanishes on the node set $\mathcal{Y}$, which is a maximal n-independent set in the curve q_{k-1}. Therefore, we obtain that $r = q_{k-1}s$, where $s \in \Pi_{n-k+1}$. Thus we have that

$$p\ell_{01}\ell_2 \cdots \ell_{n-k} = q_{k-1}s.$$

The lines $\ell_i,\ i = 2,\ldots,n-k$, are not components of q_{k-1}. Therefore, they are components of the polynomial s. Thus we obtain that

$$p\ell_{01} = q_{k-1}\beta, \text{ where } \beta \in \Pi_2.$$

Now let us verify that ℓ_{01} is a component of q_{k-1}. Indeed, otherwise it is a component of the conic β and we get that

$$p \in \Pi_k,\ p|_{\mathcal{X}} = 0 \implies p = q_{k-1}\ell, \text{ where } \ell \in \Pi_1.$$

Therefore, we get $\dim \mathcal{P}_{k,\mathcal{X}} = 3$, which contradicts the hypothesis.

Thus we conclude that

$$q_{k-1} = \ell_{01}q_{k-2}, \text{ where } q_{k-2} \in \Pi_{k-2}.$$

The curve q_{k-2} passes through at most $d(n,k-2)$ nodes from $\mathcal{X}$. Hence we get that at least 2 nodes from $\mathcal{X}$ belong to the line ℓ_{01}.

Next we will show that exactly 2 nodes from $\mathcal{X}$ belong to ℓ_{01}, which will prove Lemma. Assume by way of contradiction that at least 3 nodes from $\mathcal{X}$ lie in ℓ_{01}. First let us show that all the nodes of $\mathcal{A}$ belong to ℓ_{01}. Suppose conversely that a node from $\mathcal{A}$, say A_2, does not belong to the line ℓ_{01}. Then in the same way as in the case of the line ℓ_{01} we get that ℓ_{02} is a component of q_{k-1}. Thus the node A_0 is an intersection point of two components of q_{k-1}, i.e., ℓ_{01} and ℓ_{02}, which contradicts our assumption.

Next let us verify that in the beginning we could choose a non-collinear n-independent set $\mathcal{A} \subset q_{k-1}$, which will be a contradiction and will complete the proof. To this end let us prove that one can move any node of $\mathcal{A}$, say A_0, from ℓ_{01} to the other component q_{k-2} such that the resulted set $\mathcal{A}$ remains n-independent.

In view of Lemma 6, *ii*), for this we need to find an n-fundamental polynomial of A_0, for which q_{k-2} is not a component. Let us show that any fundamental polynomial of A_0 has this property. Indeed, suppose conversely that for an n-fundamental polynomial $p^{\star}_{A_0} \in \Pi_n$ the curve q_{k-2} is a component, i.e., $p^{\star}_{A_0} = q_{k-2}r$, where $r \in \Pi_{n-k+2}$. We get from here that r vanishes at all the nodes in $\mathcal{Y} \cap \ell_{01}$ except A_0. Thus r vanishes at $\geq 3 + (n-k+1) - 1 = n-k+3$ nodes in ℓ. Therefore, in view of Proposition 3, r vanishes at all the points of ℓ_{01} including A_0, which is a contradiction. □

Now we are in a position to present

Proof of Theorem 3. Recall that it remains to prove the direct implication. Let $\sigma_1,\ldots,\sigma_4$ be the four curves of degree $\leq k$ that pass through all the nodes of the n-independent set $\mathcal{X}$ with $|\mathcal{X}| = d(n,k-2)+2$. First we will consider

Case $n \geq k+2$. Let us start by choosing three nodes $B_1,B_2,B_3 \notin \mathcal{X}$ such that the following four conditions are satisfied:

i) the set $\mathcal{X} \cup \{B_1,B_2,B_3\}$ is n-independent;

ii) the nodes B_1,B_2,B_3 are non-collinear;

iii) each line through B_i and B_j, $1 \leq i < j \leq 3$, does not pass through any node from $\mathcal{X}$;

$iv)$ for any subset $\mathcal{A} \subset \mathcal{X}$, $|\mathcal{A}| = 3$ the set $\mathcal{A} \cup \{B_1, B_2, B_3\}$ is 2-poised.

Let us verify that one can find such nodes. Indeed, in view of Lemma 3, we can start by choosing some nodes B_i', $i = 1,2,3$, satisfying the condition $i)$. Then, according to Lemma 2, for some positive ε all the nodes in ε neighborhoods of $B_i', i = 1,2,3$, satisfy the condition $i)$. Next, by using Lemma 4, three times, for the nodes B_i', $i = 1,2,3$, consecutively, we obtain that there are nodes B_i'', $i = 1,2,3$, satisfying the condition $iv)$ and $\rho(B_i'', B_i') < (1/2)\varepsilon$, $i = 1,2,3$. Now notice that both conditions $i)$ and $iv)$ are satisfied for B_i'', $i = 1,2,3$. Then, according to Lemma 2, for some positive $\varepsilon' > 0$ all the nodes in ε' neighborhoods of B_i'', $i = 1,2,3$, satisfy the conditions $i)$ and $iv)$. Finally, from these ε' neighborhoods we can choose the nodes B_i, $i = 1,2,3$, satisfying the conditions $ii)$, $iii)$, too.

Note that, in view of Proposition 1, the condition $iv)$ means that

$v)$ any conic through the triple B_1, B_2, B_3 passes through at most two nodes from $\mathcal{X}$.

Next, in view of Proposition 5, there is a curve of degree at most k, denoted by σ, which passes through all the nodes of $\mathcal{X}' := \mathcal{X} \cup \{B_1, B_2, B_3\}$.

Now notice that the curve σ passes through more than $d(n, k-2)$ nodes and, therefore, its degree equals either to $k-1$ or k. By taking into account Lemma 7, we may assume that the degree of the curve σ equals to k. Evidently, in view of Lemma 7, we may assume also that σ has no multiple component.

Therefore, by using Proposition 6, we can extend the set $\mathcal{X}'$ till a maximal n-independent set $\mathcal{X}'' \subset \sigma$. Notice that, since $|\mathcal{X}''| = d(n,k)$, we need to add a set of $d(n,k) - (d(n,k-2)+2) - 3 = 2(n-k)$ nodes to $\mathcal{X}'$, denoted by $\mathcal{A} := \{A_1, \ldots, A_{2(n-k)}\} : \mathcal{X}'' := \mathcal{X} \cup \{B_1, B_2, B_3\} \cup \mathcal{A}$.

Thus the curve σ becomes maximal with respect to this set. In view of Lemma 6, $i)$, we require that each node of $\mathcal{A}$ may belong only to one component of the curve σ. Then, by using Lemma 5, we get a curve σ_0 of degree at most k, different from σ that passes through all the nodes of $\mathcal{X}$ and two more arbitrary nodes, which will be specified below.

We intend to divide the set of nodes $\mathcal{A}$ into $n-k$ pairs such that the lines $\ell_1, \ldots, \ell_{n-k-1}$ through $n-k-1$ pairs from them, respectively, are not components of σ. The remaining pair we associate with the curve σ_0. More precisely, we require that σ_0 passes through the two nodes of the last pair.

Before establishing the mentioned division of $\mathcal{A}$, let us verify how we can finish the proof by using it. Denote by β the conic through the triple of the nodes B_1, B_2, B_3 and the pair of nodes associated with the line ℓ_{n-k-1}. Notice that the following polynomial $\sigma_0 \beta \ell_1 \ell_2 \ldots \ell_{n-k-2}$ of degree n vanishes at all the $d(n,k)$ nodes of $\mathcal{X}'' \subset \sigma$. Consequently, according to Proposition 4, σ divides this polynomial:

$$\sigma_0 \beta \ell_1 \ell_2 \ldots \ell_{n-k-2} = \sigma q, \quad q \in \Pi_{n-k}. \tag{3}$$

The distinct lines $\ell_1, \ell_2, \ldots, \ell_{n-k-2}$ do not divide the polynomial $\sigma \in \Pi_k$, therefore, all they have to divide $q \in \Pi_{n-k}$. Therefore, we get from (3):

$$\sigma_0 \beta = \sigma \beta', \text{ where } \beta' \in \Pi_2. \tag{4}$$

Now, suppose first that the conic β is irreducible. Since the curves σ and σ_0 are different the conics β and β' also are different. Therefore, the conic β has to divide $\sigma \in \Pi_k$: $\sigma = \beta\, r, \quad r \in \Pi_{k-2}$.

Now, we derive from this relation that the curve r passes through all the nodes of the set $\mathcal{X}$ but two. Indeed, σ passes through all the nodes of $\mathcal{X}$. Therefore, these nodes are either in the curve r or in the conic β. But the latter conic passes through the triple of nodes B_1, B_2, B_3, and according to the condition v), it passes through at most two nodes of $\mathcal{X}$. Thus r passes through at least $d(n,k-2)$ nodes of $\mathcal{X}$. Since r is a curve of degree $k-2$, we conclude that r is a maximal curve and passes through exactly $d(n,k-2)$ nodes of $\mathcal{X}$.

Next suppose that the conic β is reducible. Consider first the case when the pair of nodes associated with the line ℓ_{n-k-1} is collinear with a node from the triple B_1, B_2, B_3, say with B_1. Thus we have that $\beta = \ell_{n-k-1}\ell$, where the line ℓ passes through the nodes B_2, B_3.

The line ℓ_{n-k-1} does not divide the polynomial $\sigma \in \Pi_k$, therefore it has to divide β'. Therefore we get from the relation (4) that

$$\sigma_0\,\ell = \sigma\,\ell', \text{ where } \ell' \in \Pi_2. \tag{5}$$

Now, the lines ℓ and ℓ' are different, so ℓ has to divide $\sigma \in \Pi_k$:

$$\sigma = \ell\, r, \quad r \in \Pi_{k-1}.$$

In view of above condition iii), the line ℓ does not pass through any node of $\mathcal{X}$. Therefore, the curve r of degree $k-1$ passes through all the nodes of $\mathcal{X}$. Thus the proof of Theorem is completed in view of Lemma 7.

Observe that we may conclude from here that any line component of the curve σ, as well as of the curve σ_0, passes through at least a node from $\mathcal{X}$. Thus, in view of (iii) the (three) lines through two nodes from $\{B_1, B_2, B_3\}$ are not a component of σ. Hence, in view of Lemma 6, we may assume that the nodes of $\mathcal{A}$ do not belong to these three lines. Consequently, no extra case of a reducible β is possible.

Next let us establish the above mentioned division of the node set $\mathcal{A}$ into $n-k$ pairs such that the lines $\ell_1, \ldots, \ell_{n-k-1}$ through $n-k-1$ pairs from them, respectively, are not components of σ. Thus we need to have pairs of nodes not belonging to the same line component of σ.

Recall that the nodes of $\mathcal{A}$ belong only to one component of the curve σ. Therefore, the line components do not intersect at the nodes of $\mathcal{A}$. By using induction on $n-k$, it can be proved easily that the mentioned division of $\mathcal{A}$ is possible if and only if no $n-k$ nodes of $\mathcal{A}$, not counting those two associated with the curve σ_0, are located in a line component. Observe also that any two nodes of the set $\mathcal{A}$ may be considered as associated with σ_0.

Now note that there can be at most two undesirable line components of the curve σ, each of which contains $n-k$ nodes from $\mathcal{A}$. In this case one node from each of the two components we associate with σ_0.

Suppose that there is only one undesirable line component with $n-k$ or $n-k+1$ nodes. Then one or two nodes from here we associate with σ_0, respectively.

Finally consider the case of one undesirable line component ℓ with $m \geq n-k+2$ nodes. Recall that each line component passes through at least a node from $\mathcal{X}$. We have that $\sigma = \ell q$, where $q \in \Pi_{k-1}$ is a component of σ. Now, in view of Lemma 6, *ii*), we will move $m-n+k-1$ nodes, one by one, from ℓ to the component q. For this it suffices to prove that during this process each node $A \in \ell \cap \mathcal{A}$ has no fundamental polynomial, for which the curve q is a component. Suppose conversely that $p_A^\star = qr,\ r \in \Pi_{n-k+1}$. Now we have that r vanishes at $\geq n-k+1$ nodes in $\ell \cap \mathcal{A} \setminus \{A\}$, and at least at a node from $\ell \cap \mathcal{X}$ mentioned above. Thus r together with $p_A^\star$ vanishes at the whole line ℓ, including the node A, which is a contradiction. It remains to note that there will be no more undesirable line, except ℓ, in the resulted set $\mathcal{A}$ after the described movement of the nodes, since we keep exactly $n-k+1$ nodes in $\ell \cap \mathcal{A}$. Finally let us consider

Case $\boldsymbol{n = k+1}$. Consider three collinear nodes $B_1, B_2, B_3 \notin \mathcal{X}$ such that the following two conditions are satisfied:

i′) the set $\mathcal{X} \cup \{B_1, B_2, B_3\}$ is n-independent;

ii′) the line through $B_i,\ i = 1,2,3$, does not pass through any node from $\mathcal{X}$.

Let us verify that one can find such nodes B_1, B_2, B_3, or the conclusion of Theorem 3 holds. Indeed, in view of Lemma 3, we can start by choosing some two nodes $B_i',\ i = 1,2$, such that

i″) the set $\mathcal{X} \cup \{B_1', B_2',\}$ is n-independent.

Then, according to Lemma 2, for some positive ε all the nodes in ε neighborhoods of $B_i',\ i = 1,2$, satisfy *i*″). Thus, from this neighborhoods we can choose the nodes $B_i,\ i = 1,2$, such that the line through them ℓ_0 does not pass through any node from $\mathcal{X}$. Now it remains to prove Theorem 3 under the assumption that there is no node $B_3 \in \ell_0$ such that the condition *i*′) holds.

Indeed, this means that any polynomial $p \in \Pi_n$ vanishing on $\mathcal{X} \cup \{B_1, B_2,\}$ vanishes identically on ℓ_0. In view of Lemma 5, we may choose a such polynomial p from the linear span of four linearly independent curves of the hypothesis. Then we get that $p \in \Pi_k,\ p|_{\ell_0} = 0$. Thus we have $p = \ell_0 q$, where $q \in \Pi_{k-1}$. Now, in view of *ii*′) we readily deduce that the curve q of degree $\leq k-1$ passes through all the nodes of $\mathcal{X}$. Thus the proof of Theorem is completed in view of Lemma 7.

Now we may assume that we have three collinear nodes $B_1, B_2, B_3 \notin \mathcal{X}$, satisfying the conditions *i*′) and *ii*′).

Next, as in the previous case, we get a curve of degree k, denoted by σ, which has no multiple component and passes through all the nodes of $\mathcal{X}' := \mathcal{X} \cup \{B_1, B_2, B_3\}$. Then, by using Proposition 6, we extend the set $\mathcal{X}'$ till a maximal n-independent set $\mathcal{X}'' = \mathcal{X}' \cup \mathcal{A} \subset \sigma$. Note that $|\mathcal{A}| = 2$ in this case.

Then, as in the previous case, we get a curve σ_0 of degree k different from σ, passing through all the nodes of the set $\mathcal{X}$ and two nodes of $\mathcal{A}$. Now observe that the polynomial $\sigma_0 \ell_0 \in \Pi_{k+1}$ vanishes on the maximal $n = (k+1)$-independent set $\mathcal{X}'' \subset \sigma$. Therefore we have that $\sigma_0 \ell_0 = \sigma \ell$ where $\ell \in \Pi_1$. Since σ_0 and σ are different so are also ℓ_0 and ℓ. Thus ℓ_0 is a component of σ, i.e., $\sigma = \ell_0 r$, where $r \in \Pi_{k-1}$. Now, in view of above condition *ii*′), the line ℓ_0 does not pass through any node of $\mathcal{X}$. Therefore, the curve r of degree $k-1$ passes through all the nodes of $\mathcal{X}$. Thus the proof of Theorem is completed in view of Lemma 7. □

An Application to the Gasca-Maeztu Conjecture. Recall that a node $A \in \mathcal{X}$ uses a line ℓ means that ℓ is a factor of the fundamental polynomial $p = p^\star_A$, i.e., $p = \ell r$ for some $r \in \Pi_{n-1}$.

A GC_n-set in the plane is an n-poised set of nodes, where the fundamental polynomial of each node is a product of n linear factors. The Gasca–Maeztu conjecture states that any GC_n-set possesses a subset of $n+1$ collinear nodes.

It was proved in [8], that any line passing through exactly 2 nodes of a GC_n-set $\mathcal{X}$ can be used at most by one node from $\mathcal{X}$.

It was proved in [7] that any used line passing through exactly 3 nodes of a GC_n-set $\mathcal{X}$ can be used either by exactly one or three nodes from $\mathcal{X}$.

Below we consider the case of lines passing through exactly 4 nodes.

Corollary . *Let $\mathcal{X}$ be an n-poised set of nodes and ℓ be a line, which passes through exactly* 4 *nodes. Suppose ℓ is used by at least four nodes from $\mathcal{X}$. Then it is used by exactly six nodes from $\mathcal{X}$. Moreover, if it is used by six nodes, then they form a* 2*-poised set. Furthermore, in the latter case, if $\mathcal{X}$ is a GC_n set, then the six nodes form a GC_2 set.*

Proof. Assume that $\ell \cap \mathcal{X} = \{A_1, \dots, A_4\} =: \mathcal{A}$. Assume also that the four nodes in $\mathcal{B} := \{B_1, \dots, B_4\} \in \mathcal{X}$ use the line ℓ, that is,

$$p^\star_{B_i} = \ell q_i,\ i = 1, \dots, 4, \text{ where } q_i \in \Pi_{n-1}.$$

The polynomials $q_1, \dots, q_4$ vanish at $N-8$ nodes of the set $\mathcal{X}' := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B})$. Hence through these $N-8 = d(n, n-3)+2$ nodes pass four linearly independent curves of degree $n-1$. By Theorem 3, there exists a maximal curve μ of degree $n-3$ passing through $N-10$ nodes of $\mathcal{X}'$ and the remaining two nodes denoted by C_1, C_2 are outside of it. Now, according to Proposition 5, the nodes C_1, C_2 use μ :

$$p^\star_{C_i} = \mu r_i,\ r_i \in \Pi_3,\ i = 1, 2.$$

These polynomials r_i have to vanish at the four nodes of $\mathcal{A} \subset \ell$. Hence $q_i = \ell \beta_i,\ i = 1, 2$, with $\beta_i \in \Pi_2$. Therefore, the nodes C_1, C_2 use the line ℓ :

$$p^\star_{C_i} = \mu \ell \beta_i,\ i = 1, 2.$$

Hence, if four nodes in $\mathcal{B} \subset \mathcal{X}$ use the line ℓ, then there exist two more nodes $C_1, C_2 \in \mathcal{X}$ using it and all the nodes of $\mathcal{Y} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B} \cup \{C_1, C_2\})$ lie in a maximal curve μ of degree $n-3$: $\mathcal{Y} \subset \mu$.

Next, let us show that there is no seventh node using ℓ. Assume by way of contradiction that except of the six nodes in $\mathcal{S} := \{B_1, \dots, B_4, C_1, C_2\}$, there is a seventh node D using ℓ. Of course we have that $D \in \mathcal{Y}$.

Then we have that four nodes B_1, B_2, B_3 and D are using ℓ, therefore, as it was proved above, there exist two more nodes $E_1, E_2 \in \mathcal{X}$ (which may coincide or not with B_4 or C_1, C_2) using it and all the nodes of $\mathcal{Y}' := \mathcal{X} \setminus (\mathcal{A} \cup \{B_1, B_2, B_3, D, E_1, E_2\})$ lie in a maximal curve μ' of degree $n-3$. We have also that

$$p^\star_D = \mu' q',\ q' \in \Pi_3. \tag{6}$$

Now, notice that both the curves μ and μ' pass through all the nodes of the set $\mathcal{Z} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B} \cup \{C_1, C_2, D, E_1, E_2, \})$ with $|\mathcal{Z}| \geq N-13$.

Then, we get from Theorem 1 with $k = n-4$, that $N-13 = d(n,n-4)+2$ nodes determine the curve of degree $n-3$ passing through them uniquely. Thus μ and μ' coincide. Therefore, in view of $\mathcal{Y} \subset \mu$ and (6), $p_D^\star$ vanishes at all the nodes of $\mathcal{Y}$, which is a contradiction since $D \in \mathcal{Y}$.

Now let us verify the last “moreover" statement. Suppose the six nodes in $\mathcal{S} \subset \mathcal{X}$ use the line ℓ. Then, as we obtained earlier, the nodes $\mathcal{Y} := \mathcal{X} \setminus (\mathcal{A} \cup \mathcal{B} \cup \{C_1, C_2\})$ are located in a maximal curve μ of degree $n-3$. Therefore, the fundamental polynomial of each $A \in \mathcal{S}$ uses $\mu:\ p_A^\star = \mu q_A$, where $q_A \in \Pi_2$. It is easily seen that q_A is a 2-fundamental polynomial of $A \in \mathcal{S}$. $\square$

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Հ. Ա. ՀԱԿՈԲՅԱՆ, Հ. Մ. ՔԼՈՅԱՆ

n-ԱՆԿԱԽ ՀԱՆԳՈՒՅՑՆԵՐՈՎ ԱՆՑՆՈՂ ՀԱՆՐԱՀԱՇՎԱԿԱՆ ԿՈՐԵՐԻ ՏԱՐԱԾՈՒԹՅՈՒՆՆԵՐԻ ՉԱՓՈՂԱԿԱՆՈՒԹՅԱՆ ՎԵՐԱԲԵՐՅԱԼ

Դիցուք $\mathcal{X}$-ը հարթության վրա n-անկախ հանգույցների բազմություն է, այսինքն՝ յուրաքանչյուր հանգույց ունի n աստիճանի ֆունդամենտալ բազմանդամ: Ենթադրենք, որ $|\mathcal{X}| = (n+1)+n+\cdots+(n-k+4)+2$ և $3 \le k \le n-1$: Հոդվածում ապացուցում ենք, որ կարող են լինել k-ից փոքր կամ հավասար աստիճանի ամենաշատը 4 գծորեն անկախ կորեր, որոնք անցնում են $\mathcal{X}$-ի բոլոր հանգույցներով: Մենք տալիս ենք այն դեպքի բնութագիրը, երբ կա այդպիսի ճիշտ 4 կոր: Այն է, մենք ապացուցում ենք, որ այդ դեպքում $\mathcal{X}$ բազմությունն ունի շատ հատուկ կառուցվածք՝ բոլոր հանգույցները, բացի երկուսից, պատկանում են $k-2$ աստիճանի (մաքսիմալ) կորի: Վերջում Գասքա-Մաեզթուի վարկածի համար բերվում է մի կարևոր կիրառություն: