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# THE NON-CLASSICAL PROBLEM OF AN ORTHOTROPIC BEAM OF VARIABLE THICKNESS WITH THE SIMULTANEOUS ACTION OF ITS OWN WEIGHT AND COMPRESSIVE AXIAL FORCES

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On the basis of the refined theory of orthotropic plates of variable thickness, the equations of the problem of bending of an elastically clamped beam in the case of simultaneous action of its own weight and axial compressive forces are obtained. The effects of transverse shear and the effect of reducing the compressive force of the support are taken into account. Turning to dimensionless quantities, the specific problem for a beam of linearly variable thickness is solved by the collocation method. The question of the stability of the beam is discussed.

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***Keywords*:** elastically fastened support, bending, transverse shear, sole weight, axial compressive forces, stability.

**Introduction.** While designing building structures and devices, situations often arise when it is necessary to solve the problem of bending of thin-walled elements under the action of axial compressive forces and their own weight. There are many papers in which the problems of intense deformation of the state and stability of such elements are investigated within the framework of the classical theory of mechanics (see [1]). The advent of modern materials has led to the need for the mentioned research on refined theories that take into account those factors that are neglected in the classical theory. This paper attempts to partially fulfill this gap.

**Theoretical Part.** Consider an orthotropic beam of length $l$, constant width $b$ and variable thickness $h$ in the right-hand Cartesian coordinate system $x$, $y$, $z$. The main directions of the anisotropy of the material are parallel to the coordinate axes. The beam is elastically clamped at two ends and, in addition of its own weight, is also affected by axial compressive forces $T$ (see Fig. 1). It is taken into account that the

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elastically clamped support due to friction with an elastic array reduces the external compressive force $P$, as a result of which the force

$$T=\beta P, \quad \beta<1, \tag{1}$$

acts on the beam. The value of the coefficient $\beta$ can be easily determined experimentally.

In the paper [2] the conditions of the considered elastically clamped support with a transverse bending of the beam were obtained. These conditions are:

$$\frac{dw}{dx}=D\left(aN_x-M_x\right), \quad w=a\frac{dw}{dx}+BN_x, \tag{2}$$

where $w$ is the deflection; $N_x$ and $M_x$ are the transverse force and bending moment of the beam, respectively; $D$ and $B$ are parameters of the elastically clamped support, and are inverse quantities of the stiffness of the support against rotation and vertical displacement, respectively. In the SI units they have dimensions $D\sim N^{-1}m^{-1}$, $B\sim mN^{-1}$, respectively. Parameters $D$ and $B$ are connected by the relation

$$D=\frac{3B}{a^2}. \tag{3}$$

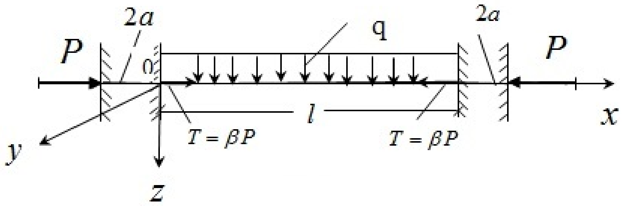


Fig. 1. The loading form.

Note that in the derivation of conditions (2) it was assumed that, due to the relative smallness of length $2a$, a part of the beam inserted into an elastic mass moves without deformation progressively and rotates as one piece, in virtue of which the derivative $\frac{dw}{dx}$ of the deflection within the inserted parts do not change and are equal to the values at $x=0$ and $x=l$.

Using the refined theory of orthotropic plates of variable thickness (see [3]), we obtain the following differential equations for the bending problem of the beam under consideration:

$$\begin{cases}\left(Ebh^2\dfrac{d^2h}{dx^2}+12\beta P\right)\dfrac{d^2w}{dx^2}-bh\left(8+\chi h\dfrac{d^2h}{dx^2}\right)\dfrac{d\varphi_1}{dx}-16b\dfrac{dh}{dx}\varphi_1=12\rho gbh,\\ Eh^2\dfrac{d^3w}{dx^3}+2Eh\dfrac{dh}{dx}\cdot\dfrac{d^2w}{dx^2}-\chi h^2\dfrac{d^2\varphi_1}{dx^2}-2\chi h\dfrac{dh}{dx}\cdot\dfrac{d\varphi_1}{dx}+8\varphi_1=0.\end{cases} \tag{4}$$

Due to the lack of stresses $\sigma_y$ and neglecting of voltage $\sigma_z$, we have replaced material parameter $B_{11}$ by Young's modulus $E$ of axial direction, $\chi$ takes into account the effect of transverse shear deformation $e_{xz}$, $\varphi_1$ is a function characterizing the distribution of tangential stress $\tau_{xz}$ in the midplane $z=0$ of the beam, $\rho$ denotes the density of the beam material, and $g$ is the gravity acceleration.

Note that in the expression of the cargo term $Z_2$ [4], the intensity of a fictitious load resulting from the compression of a curved beam by forces $\beta P$ is added to the intensity of the vertical load arising from its own weight.

Thus, the problem we are considering reduces to solving the system of differential equations (4) under the boundary conditions (2) imposed on the both edges of the beam, i.e., $x=0$ and $x=l$.

For simplicity, we assume that both edges of the beam have the same elastically pinched supports.

Let us apply the dimensionless notation

$$\begin{gathered} x=l\bar{x}, \quad h=m_1 lH, \quad b=m_2 l, \quad a=m_1 l, \quad \varphi_1=E\overline{\varphi}, \\ P=Em_1^2 l^2\overline{P}, \quad \rho=\frac{Em_1^3\overline{\rho}}{m_2 gl}, \quad D=\frac{3\overline{B}}{Em_1^2 l^3}, \quad w=a\overline{w}, \quad BEl=\overline{B}. \end{gathered} \tag{5}$$

In the problem, we are considering, expressions for the shear force of the beam $N_x$ and the bending moment $M_x$ can be obtained from the corresponding expressions for the case of the plate [3], multiplying the letter by the width of the beam $b$. In this way we get:

$$\begin{gathered} N_x=\frac{bh}{12}\left[8\varphi_1-h\frac{dh}{dx}\left(E\frac{d^2w}{dx^2}-\chi\frac{d\varphi_1}{dx}\right)\right], \\ M_x=-\frac{bh^3E}{12}\left(\frac{d^2w}{dx^2}-a_{55}\frac{d\varphi_1}{dx}\right). \end{gathered} \tag{6}$$

In view of the notation (5) the equations (4) take the form:

$$\begin{cases} m_1^2\left(m_1m_2H^2\dfrac{d^2H}{d\bar{x}^2}+12\beta\overline{P}\right)\dfrac{d^2\overline{w}}{d\bar{x}^2}-m_2H\left(8+\chi m_1^2H\dfrac{d^2H}{d\bar{x}^2}\right)\dfrac{d\overline{\varphi}}{d\bar{x}}- \\ \qquad -16m_2\dfrac{dH}{d\bar{x}}\overline{\varphi}=12m_1^3\overline{\rho}H, \\ m_1^3H^2\dfrac{d^3\overline{w}}{d\bar{x}^3}+2m_1^3H\dfrac{dH}{d\bar{x}}\cdot\dfrac{d^2\overline{w}}{d\bar{x}^2}-\chi m_1^2H^2\dfrac{d^2\overline{\varphi}}{d\bar{x}^2}-2\chi m_1^2H\dfrac{dH}{d\bar{x}}\cdot\dfrac{d\overline{\varphi}}{d\bar{x}}+ \\ \qquad +8\overline{\varphi}=0. \end{cases} \tag{7}$$

The boundary conditions (2), which are imposed on both edges of the beam, in view of (6), take the form:

$$\begin{gathered} \frac{d\overline{w}}{d\bar{x}}=\frac{\overline{B}m_2H}{4m_1}\left[8\overline{\varphi}+m_1H\left(H-m_1\frac{dH}{d\bar{x}}\right)\left(m_1\frac{d^2\overline{w}}{d\bar{x}^2}-\chi\frac{d\overline{\varphi}}{d\bar{x}}\right)\right], \\ \overline{w}=m_1\frac{d\overline{w}}{d\bar{x}}+\frac{\overline{B}m_2H}{12m_1}\left[8\overline{\varphi}-m_1^2H\frac{dH}{d\bar{x}}\left(m_1\frac{d^2\overline{w}}{d\bar{x}^2}-\chi\frac{d\overline{\varphi}}{d\bar{x}}\right)\right], \end{gathered} \tag{8}$$

at $\bar{x}=0$ and $\bar{x}=1$.

Consider the case when the thickness of the beam varies linearly with $x$:

$$h=m_1l+h_1x=m_1lH \quad \Rightarrow \quad H=1+\gamma\bar{x}, \tag{9}$$

where

$$\gamma = \frac{h_1}{m_1}, \quad h > 0 \quad \Rightarrow \quad h_1 > -m_1. \tag{10}$$

With this specification, the equations (7) take the form:

$$\begin{cases} 3\beta m_1^2 \overline{P} \dfrac{d^2\overline{w}}{d\overline{x}^2} - 2m_2 (1+\gamma\overline{x}) \dfrac{d\overline{\varphi}}{d\overline{x}} - 4m_2\gamma\overline{\varphi} = 3m_1^3 \overline{\rho} (1+\gamma\overline{x}), \\ m_1^3 (1+\gamma\overline{x})^2 \dfrac{d^3\overline{w}}{d\overline{x}^3} + 2m_1^3\gamma(1+\gamma\overline{x}) \dfrac{d^2\overline{w}}{d\overline{x}^2} - \\ \quad -\chi m_1^2 (1+\gamma\overline{x})^2 \dfrac{d^2\overline{\varphi}}{d\overline{x}^2} - 2\chi m_1^2\gamma(1+\gamma\overline{x}) \dfrac{d\overline{\varphi}}{d\overline{x}} + 8\overline{\varphi} = 0, \end{cases} \tag{11}$$

and the boundary conditions (8) take the form:

$$\begin{gathered} \frac{d\overline{w}}{d\overline{x}} = \frac{\overline{B}m_2 (1+\gamma\overline{x})}{4m_1} \left[ 8\overline{\varphi} + m_1 (1+\gamma\overline{x}) (1+\gamma\overline{x} - m_1\gamma) \left( m_1 \frac{d^2\overline{w}^2}{d\overline{x}} - \chi \frac{d\overline{\varphi}}{d\overline{x}} \right) \right], \\ \overline{w} = m_1 \frac{d\overline{w}}{d\overline{x}} + \frac{\overline{B}m_2 (1+\gamma\overline{x})}{12} \left[ 8\overline{\varphi} - m_1^2\gamma(1+\gamma\overline{x}) \left( m_1 \frac{d^2\overline{w}}{d\overline{x}^2} - \chi \frac{d\overline{\varphi}}{d\overline{x}} \right) \right], \end{gathered} \tag{12}$$

at $\overline{x} = 0$ and $\overline{x} = 1$.

**Computational Part.** Let

$$\begin{gathered} m_1 = 0.1, \quad m_2 = 0.3, \quad \gamma = 0; \quad \text{and} \quad 1, \quad \chi = 0; \quad 5 \quad \text{and} \quad 10, \\ \overline{\rho} = 0.012 \quad \text{and} \quad 0.006, \quad \beta = 0.5, \quad \overline{B} = 1. \end{gathered} \tag{13}$$

The task is convenient to solve by the collocation method. To this end, let us represent the unknown functions $\overline{w}$ and $\overline{\varphi}$ in the form of polynomials:

$$\overline{w} = a_0 + \sum_{i=1}^{k} a_i \overline{x}^i, \qquad \overline{\varphi} = b_0 + \sum_{i=1}^{k} b_i \overline{x}^i. \tag{14}$$

To determine the coefficients $a_0$, $a_i$ and $b_0$, $b_i$ we divide the interval $0 \leq \overline{x} \leq 1$ into $k$ equal parts. Considering the equations (11) at the dividing points and the boundary conditions (12) at the endpoints, we obtain a system of $2(k+1)$ algebraic equations with respect to the mentioned coefficients. Solving this system, we find the values of this coefficients, with which we calculate the values of the functions $\overline{w}$ and $\overline{\varphi}$. At the endpoints and dividing points of the segment $0 \leq \overline{x} \leq 1$, the dimensionless shear force $\overline{N}_x$ and the bending moment $\overline{M}_x$ are calculated by the corresponding formulas (5). These calculations will be repeated with increasing $k$ in the expressions (14) until the practical convergence of the calculated values $\overline{w}$, $\overline{N}_x$ and $\overline{M}_x$.zz

The calculation results are shown in the Tabs. $1-3$ and in the Figs. $2-4$ for $\gamma = 0$ and $\gamma = 1$, respectively.

*Table 1*

*The value critical of the compressive force ($\gamma$=0; $\overline{\rho}$=0.012)*

| $\chi$ | 0 | 5 | 10 |
|---|---|---|---|
| $\overline{P}_{cr.}$ | 0.1945 | 0.1578 | 0.1318 |

*Table 2*

*The deflection maximum ($\overline{w}_{max}\cdot 10^2$) depending on $\overline{P}/\overline{P}_{cr.}$ at $\gamma$=0 and different values of $\chi$ (deflection point is at $\overline{x}_{max}$=0.5)*

| | $\overline{P}/\overline{P}_{cr.}$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $\chi$=0 | 0.2641 | 0.2923 | 0.3270 | 0.3724 | 0.4320 | 0.5150 | 0.6400 | 0.8400 | 1.2300 | 2.3300 | 22.720 |
| $\chi$=5 | 0.2645 | 0.2930 | 0.3280 | 0.3738 | 0.4344 | 0.5190 | 0.6459 | 0.8568 | 1.2780 | 2.5360 | 613.00 |
| $\chi$=10 | 0.2649 | 0.2932 | 0.3285 | 0.3739 | 0.4343 | 0.5186 | 0.6449 | 0.8549 | 1.2740 | 2.5260 | 716.00 |

*Table 3*

*The deflection values ($\overline{w}\cdot 10^3$) at different points ($\overline{N}_x = 0$; $\overline{M}_x = 0$)*

| | | $\overline{x}$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $\gamma$=0 | $\chi$=0 | 0.14 | 2.26 | 4.93 | 4.77 | 1.95 | 0.06 |
| $\rho$=0.012 | $\chi$=5 | 0.17 | 2.93 | 6.54 | 6.35 | 2.57 | 0.07 |
| $\overline{P}$=0.1 | $\chi$=10 | 0.19 | 4.27 | 9.79 | 9.54 | 3.82 | 0.16 |
| $\gamma$=1 | $\chi$=0 | 0.049 | 0.46 | 0.7 | 0.52 | 0.17 | 0.032 |
| $\rho$=0.006 | $\chi$=5 | 0.055 | 0.51 | 0.76 | 0.54 | 0.17 | 0.047 |
| $\overline{P}$=0.1 | $\chi$=10 | 0.059 | 0.57 | 0.839 | 0.588 | 0.18 | 0.057 |

It should be noted that in the scientific literature there are many works devoted to the description and application of the collocation method, as well as to the study of the bending and stability of thin-walled elements with different boundary conditions, including the case of an elastically clamped support [5–16].

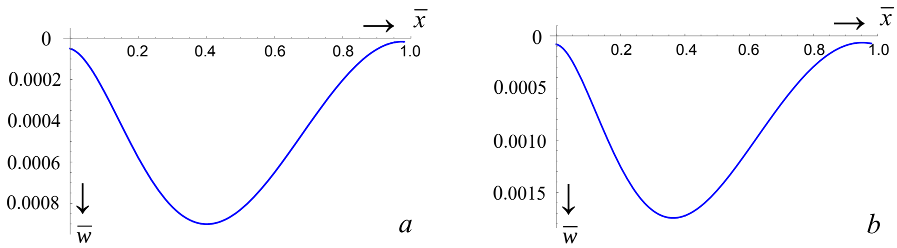


Fig. 2. The deflection form ($\gamma = 1, \overline{P} = 0.2$)
$a: \chi = 0, \quad b: \chi = 10.$

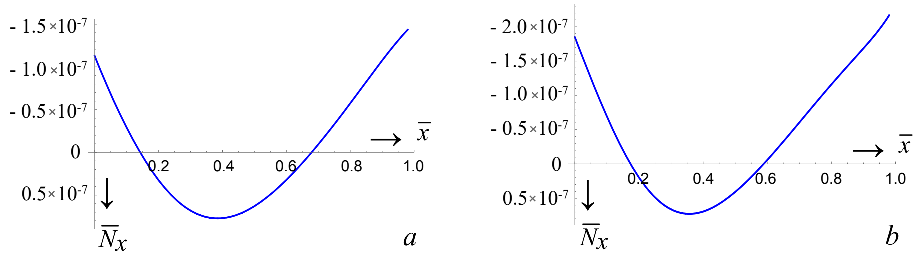


Fig. 3. The change in internal effort ($\gamma = 1$, $\overline{P} = 0.2$)
$a : \chi = 0$, $b : \chi = 10$.

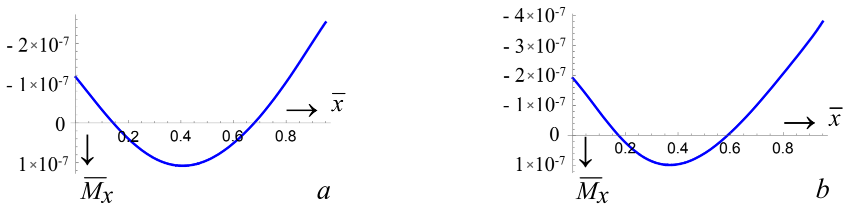


Fig. 4. The moment change ($\gamma = 1$, $\overline{P} = 0.2$)
$a : \chi = 0$, $b : \chi = 10$.

Now consider two beams made of the same isotropic elastic material. The beams are elastically clamped at both ends and are compressed by axial forces $\beta P$ (Fig. 5). One of these beams (Fig. 5,a) has a constant thickness $h_0$, and the other (Fig. 5,b) has a variable thickness $h = h_0 + h_1 x$.

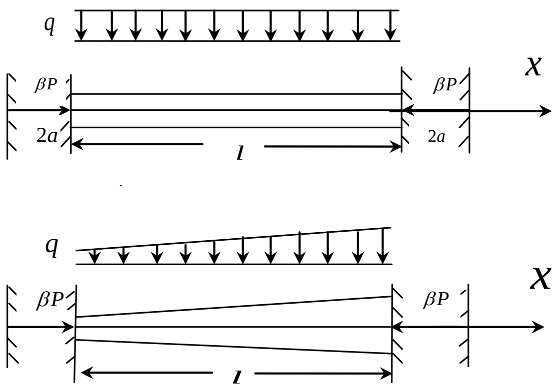


Fig. 5. The weight distribution form.

The width $b$ and length $l$ of both beams are the same. Elastically clamped supports of both beams and the reduction of the compressive forces are also identica. As a vertical load we will consider the dead weights of the beams with the intensity

$q = bh\rho g(Newton/Meter)$, where $\rho$ is the beam material density; $g$ is the acceleration of gravity (force of gravity).

It is known [1], that when the value of the compressive forces increases and approaches a critical value, then the maximum value of the deflection of a beam of constant thickness tends to infinity (case a).

In the case of a beam of variable thickness (case b), the thicknesses of the sections increase. The intensity of the own weight of this beam is greater than that of the beam of constant thickness in accordance with the law of direct proportionality $(q = bh\rho g)$, and bending stiffness $\left(\frac{Ebh^3}{12}\right)$ grows in direct proportion to the cube of thickness. Since the value of the deflection is inversely proportional to the stiffness, then if the values of the compressive forces $\beta P$ tend to be critical, then values of the deflection, in contrast to the case of the beam of constant thickness, do not tend to infinity, but to a finite value. This result is illustrated in this article when solving a specific problem of stability of a beam of variable thickness.

**Conclusion.** The data we have obtained in this article allow us to draw the following conclusions:

1. the effects of transverse shear deformations (cases $\chi > 0$), as expected, with the same values of the other quantities, leads to an increase in deflections and a decrease in critical forces;

2. the effect of transverse shear deformations does not significantly affect the character of changes in the value of the transverse force $N_x$ and bending moment $M_x$.

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## ՓՈՓՈԽԱԿԱՆ ՀԱՍՏՈՒԹՅԱՆ ՕՐԹՈՏՐՈՊ ՀԵԾԱՆԻ ԾՌՄԱՆ ՈՉ ԴԱՍԱԿԱՆ ԽՆԴԻՐԸ ՍԵՓԱԿԱՆ ԿՇՌԻ ԵՎ ԱՌԱՆՑՔԱՅԻՆ ՍԵՂՄՈՂ ՈՒԺԵՐԻ ՀԱՄԱՏԵՂ ԱԶԴԵՑՈՒԹՅԱՆ ԴԵՊՔՈՒՄ

Փոփոխական հաստության օրթոտրոպ սալերի ճշգրտված տեսության հիման վրա ստացվում են առաձգական ամրակցված հեծանի ծռման խնդրի հավասարումներն առանցքային սեղմող ուժերի և սեփական կշռի համատեղ ազդեցության դեպքում: Հաշվի են առնվում ընդլայնական սահքի և հենարանի կողմից սեղմող ուժի փոքրացման ազդեցությունները: Անցնելով անչափ մեծություններին, կոլոկացիաների եղանակով լուծվում է կոնկրետ խնդիր՝ գծայնորեն փոփոխական հաստության հեծանի համար: Քննարկվում է հեծանի կայունության հարցը: